Crux Mathematicorum

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CONTENTS

The Olympiad Corner: No. 121 ........................................ R.E. Woodrow 1

Mini-Reviews ............................................................... Andy Liu 11

Problems: 1601-1610 .......................................................... 13

Solutions: 689, 1411, 1432, 1481-1485, 1487-1491 .......................... 15

Call For Papers—ICME ...................................................... 32
THE OLYMPIAD CORNER

No. 121

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Another year has passed and with its end we see some changes in the set-up of Crux. As of this issue, as well as continuing the Corner, I shall be attempting to assist Bill with some of the work of editing and setting the journal. We will also be joined by a board of associate editors. Laurie Loro has decided that #0Cv ey ears of word processing is enough, and with some trepidation we are switching over to \LaTeX. My thanks to Laurie for all her efforts.


It is perhaps appropriate that the first set of problems of the new year be New Year’s problems from China. My thanks to Andy Liu, University of Alberta for translating and forwarding them. They were published in the Scientific Daily, Beijing.

1980 CELEBRATION OF CHINESE NEW YEAR CONTEST

February 8, 1980

1. \(ABCD\) is a rhombus of side length \(a\). \(V\) is a point in space such that the distances from \(V\) to \(AB\) and \(CD\) are both \(d\). Determine in terms of \(a\) and \(d\) the maximum volume of the pyramid \(VABCD\).

2. Let \(n\) be a positive integer. Is the greatest integer less than \((3 + \sqrt{7})^n\) odd or even?

3. A convex polygon is such that it cannot cover any triangle of area \(1/4\). Prove that it can be covered by some triangle of area \(1\).

4. Denote by \(a_n\) the integer closest to \(\sqrt{n}\). Determine

\[\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{1980}}.\]
5. A square is divided into \( n^2 \) equal squares and the diagonals of each little square are drawn. Determine, in terms of \( n \), the total number of isosceles right-angled triangles of all sizes.

1981 CELEBRATION OF CHINESE NEW YEAR CONTEST
January 26, 1981

1. What is the coefficient of \( x^2 \) when

\[
(\cdots(((x-2)^2-2)^2-2)^2-2)^2-\cdots-2)^2
\]

is expanded and like terms are combined?


3. Let \( f(x) = x^{99} + x^{98} + x^{97} + \cdots + x^2 + x + 1 \). Determine the remainder when \( f(x^{100}) \) is divided by \( f(x) \).

4. The base of a tetrahedron is a triangle with side lengths 8, 5 and 5. The dihedral angle between each lateral face and the base is 45°. Determine the volume of the tetrahedron.

5. \( ABC \) is a triangle of area 1. \( D, E \) and \( F \) are the midpoints of \( BC, CA \) and \( AB \), respectively. \( K, L \) and \( M \) are points on \( AE, CD \) and \( BF \), respectively. Prove that the area of the intersection of triangles \( DEF \) and \( KLM \) is at least 1/8.

* * * * * * *

The first solutions we print this month were in response to the challenge for the 1984 unsolved problems from Crux.


Determine the maximum area of the convex hull of four circles \( C_i \), \( i = 1, 2, 3, 4 \), each of unit radius, which are placed so that \( C_i \) is tangent to \( C_{i+1} \) for \( i = 1, 2, 3 \).

Solution by R.K. Guy, The University of Calgary.

The convex hull is dissected into two parts by the common tangent at the point of contact of \( C_2 \) and \( C_3 \). If the area of either part is less than that of the other then move \( C_1 \) (or \( C_4 \)) so as to increase the area of that part. So we may assume the symmetry of either of Figure 1 or Figure 2.

![Figure 1](image1.png)

![Figure 2](image2.png)
In each figure, the area of the convex hull is made up of sectors of each of the 4 circles (total area $\pi$), 4 rectangles and either a parallelogram (Figure 1) or an equilateral trapezium (Figure 2). So the area is either

$$\pi + 2 + 2 + 4 \cos \alpha + 4 \cos \alpha + (4 \cos \alpha)(2 \sin \alpha) = \pi + 4 + 8 \cos \alpha(1 + \sin \alpha)$$

or

$$\pi + 2 + 2 + 2 + (2 + 4 \cos \alpha) + \frac{1}{2}(2 + 2 + 4 \cos \alpha)(2 \sin \alpha)$$

$$= \pi + 4 + 4(1 + \cos \alpha)(1 + \sin \alpha).$$

Clearly the latter yields the larger area (with equality if $\cos \alpha = 1$, i.e., $\alpha = 0$).

Now $(1 + \cos \alpha)(1 + \sin \alpha)$, where $0 \leq \alpha \leq \pi/2$, is a maximum either when $\alpha$ is at one end of its domain [at $0$, $\alpha = \pi/2$ each yields $\pi + 4 + 4 \cdot 2 = \pi + 12$], or when $\cos \alpha = \sin \alpha + \cos^2 \alpha - \sin^2 \alpha = 0$, i.e. $\cos \alpha = \sin \alpha$ or $1 + \cos \alpha + \sin \alpha = 0$. Only the former gives a solution in the domain, namely $\alpha = \pi/4$. This gives

$$\pi + 4 + 4(1 + 1/\sqrt{2})^2 = \pi + 4 + 2(3 + 2\sqrt{2}) = \pi + 10 + 4\sqrt{2} > \pi + 12.$$

So $\alpha = \pi/4$ in Figure 2 yields the maximum area $\pi + 10 + 4\sqrt{2}$.

Similarly, for the minimum we must look at $\pi + 4 + 8 \cos \alpha(1 + \sin \alpha)$, where $0 \leq \alpha \leq \pi/3$. Of course, $\alpha = 0$ gives $\pi + 12$. But $\alpha = \pi/3$ gives $\pi + 8 + 2\sqrt{3}$ which is less. After differentiating, $-\sin \alpha - \sin^2 \alpha + \cos^2 \alpha = 0$, so $2\sin^2 \alpha + \sin \alpha = 1$ and $\sin \alpha = 1/2$. For $\alpha = \pi/6$, we have $\pi + 4 + 6\sqrt{3}$ which is a local maximum. Thus the minimum area occurs when $\alpha = \pi/3$ in Figure 1.

* 


Determine the maximum volume of a tetrahedron whose six edges have lengths 2, 3, 3, 4, 5, and 5.

Solution by Richard K. Guy, University of Calgary.

A triangle of sides 2, 3, 5 is degenerate; moreover there is no point at distances 3, 4, 5 from the vertices of such a triangle. The only possible triangular faces containing edge 2 are thus 233, 234, 245, 255.

<table>
<thead>
<tr>
<th>The 2 triangular faces which share edge 2 are</th>
<th>with opposite edge</th>
<th>and the other 2 triangular faces are</th>
</tr>
</thead>
<tbody>
<tr>
<td>233 &amp; 245</td>
<td>5</td>
<td>345 &amp; 355</td>
</tr>
<tr>
<td>233 &amp; 255</td>
<td>4</td>
<td>345 &amp; 345</td>
</tr>
<tr>
<td>234 &amp; 255</td>
<td>3</td>
<td>335 &amp; 345</td>
</tr>
</tbody>
</table>

Therefore the only possible tetrahedra are those shown in the following figures.
The volume of the first tetrahedron is smaller than the volume of the second, because in Fig.1 the 4-edge is not perpendicular to the plane of the 233-triangle, whereas it is perpendicular in Fig.2.

To see that the volume of the third (Fig.3) is also smaller than that of the second, take the 345-triangle as base. Then, in Fig.3, the angle that the other 3-edge makes with the 345-triangle is less than $\beta$, which is less than $\gamma$, the angle the 3-edge makes with the 345-triangle in Fig.2, since $\cos \beta = 7/9 < 7/8 = \cos \gamma$.

Thus the maximum volume occurs for the second tetrahedron and is easily calculated to be

$$\frac{1}{3} \cdot \frac{1}{2} (2\sqrt{3^2 - 1^2}) \cdot 4 = \frac{8}{3} \sqrt{2}. $$


Determine

$$\min_{A,B} \max_{0 \leq x \leq 3\pi/2} |\cos^2 x + 2 \sin x \cos x - \sin^2 x + Ax + B|. $$

Solution by Richard K. Guy, University of Calgary.

Note that

$$\cos^2 x + 2 \sin x \cos x - \sin^2 x + Ax + B = \cos 2x + \sin 2x + Ax + B = \sqrt{2} \cos(2x - \pi/4) + Ax + B. $$

The first term, $\sqrt{2} \cos(2x - \pi/4)$, oscillates between $\pm \sqrt{2}$, attaining its maximum and minimum values at $\pi/8, 5\pi/8$, and $9\pi/8$ in the domain $[0, 3\pi/2]$. Any values of $A$ and $B$ other than zero will disturb the symmetry between $\pm \sqrt{2}$ and increase the absolute value to something greater than $\sqrt{2}$. The required minimum is thus $\sqrt{2}$.

*  


N being the set of natural numbers, for every $n \in N$, let $P(n)$ denote the product of all the digits of $n$ (in base ten). Determine whether or not the sequence $\{x_k\}$, where

$$x_1 \in N, \ x_{k+1} = x_k + P(x_k), \ k = 1, 2, 3, \ldots, $$

can be unbounded (i.e., for every number $M$, there exists an $x_j$ such that $x_j > M$).
Solution by Richard K. Guy, University of Calgary.

For a $d$-digit number $n$, $P(n) \leq 9^d < 10^{d-1}$ for $d \geq 22$. So for $n > 10^{21}$, the left
hand digit of $x_k$ increments by at most 1 at each step, so that the leftmost digits will eventu-
ally be 10\ldots and $P(x_k) = 0$ from then on, and the sequence becomes constant.

It’s of interest, perhaps, to ask for the sequence with the largest number of distinct
entries. Is it easy to beat $1, 2, 4, 8, 16, 22, 26, 38, 62, 74, 102, \ldots$?

*  


A unit square is to be covered by three congruent disks.

(a) Show that there are disks with radii less than half the diagonal of the square
that provide a covering.

(b) Determine the smallest possible radius.

Solution by Richard K. Guy, University of Calgary.

(a) In the covering shown, the radius of the up-
ner disks are

$$\frac{1}{2}\sqrt{(1/2)^2 + (7/8)^2} = \frac{\sqrt{65}}{16}.$$ 

and the radius of the lower disk is

$$\frac{1}{2}\sqrt{1^2 + (1/8)^2} = \frac{\sqrt{65}}{16},$$

and $\sqrt{65}/16 < \sqrt{2}/2$.

(b) Suppose there is a covering with three disks,
each of diameter less than $\sqrt{65}/8$. Two of the four cor-
ners are covered by the same disk, by the pigeon-hole
principle. Without loss of generality we can assume
that they are $A$ and $B$. Then the point $E$ is covered
by a different disk, since $EB = \sqrt{65}/8$. And the point
$G$ is covered by the third disk since $EG = \sqrt{65}/8$ and
$GB > \sqrt{65}/8$. This means that the point $F$ is covered
by the second disk, since $FA = FG = \sqrt{65}/8$. Then the
point $K$, where $FK = 1/8$, is covered by the third disk,
since $KA > KE = \sqrt{65}/8$. This means $D$ is uncovered, since $DB > DF > DK > \sqrt{65}/8$.
This contradicts our assumption that the three disks cover the square. So $\sqrt{65}/8$ is the
least possible radius.

*  


For $0 \leq r \leq n$, let $a_n$ be the number of binomial coefficients \binom{n}{r} which leave remain-
der 1 on division by 3, and let $b_n$ be the number which leave remainder 2. Prove
that $a_n > b_n$ for all positive integers $n$. 

**Solution by Andy Liu, University of Alberta.**

All congruences (≡) are taken modulo 3, and we say that a polynomial is *satisfactory* if it has more coefficients congruent to 1 than it has congruent to 2. Let

\[ n = 3^k n_k + 3^{k-1} n_{k-1} + \cdots + 3 n_1 + n_0 \]

be the base 3 representation of \( n \), where \( n_i = 0, 1 \) or 2 for \( 0 \leq i \leq k \). Then we have that

\[
(1 + x)^n = (1 + x)^{3^k n_k} (1 + x)^{3^{k-1} n_{k-1}} \cdots (1 + x)^{3 n_1} (1 + x)^{n_0} \\
= (1 + x^{3^k})^{n_k} (1 + x^{3^{k-1}})^{n_{k-1}} \cdots (1 + x^3)^{n_1} (1 + x)^{n_0}.
\]

For \( 0 \leq i \leq k \), set

\[ F_i(x) = (1 + x^{3^i})^{n_i} (1 + x^{3^{i-1}})^{n_{i-1}} \cdots (1 + x^3)^{n_1} (1 + x)^{n_0}. \]

We claim that \( F_i(x) \) is satisfactory for \( 0 \leq i \leq k \). This is certainly true for \( i = 0 \). Suppose it holds for some \( i < k \), and consider \( F_{i+1}(x) = (1 + x^{23^i}) F_i(x) \).

If \( n_i + 1 = 0 \), we have \( F_{i+1}(x) = F_i(x) \), and the result follows from the induction hypothesis.

If \( n_i + 1 = 1 \), then \( F_{i+1}(x) = F_i(x) + x^{23^i} F_i(x) \). By the induction hypothesis \( F_i(x) \) is satisfactory, and so is \( x^{23^i} F_i(x) \). Moreover, since \( F_i(x) \) is of degree strictly less than \( 3^{i+1} \), there are no like terms between \( F_i(x) \) and \( x^{23^i} F_i(x) \). It follows that \( F_{i+1}(x) \) is also satisfactory.

If \( n = 2 \), then \( F_{i+1}(x) = F_i(x) + 2 x^{3^i+1} F_i(x) + x^{23^i+1} F_i(x) \). Again there are no like terms. Moreover, the numbers of coefficients of \( 2 x^{3^i+1} F_i(x) \) congruent to 1 and 2 are respectively equal to the numbers of coefficients of \( F_i(x) \) congruent to 2 and 1. Hence \( F_{i+1}(x) \) is satisfactory.

It follows that \( (1 + x)^n \equiv F_k(x) \) is satisfactory, so that \( a_n > b_n \) as desired.

* *


If \( G \) is a multiplicative group and \( a, b, c \) are elements of \( G \), prove:

(a) If \( b^{-1} a b = a c \), \( a c = c a \), and \( b c = c b \), then \( a^n b = a^n c^n \) and \( (ab)^n = b^n a^n c^{n(n+1)/2} \) for all \( n \in \mathbb{N} \).

(b) If \( b^{-1} a b = a^k \) where \( k \in \mathbb{N} \), then \( b^{-1} a^n b^l = a^{nk} \) for all \( l, n \in \mathbb{N} \).

**Solution by Richard K. Guy, University of Calgary.**

(a) First \( b^{-1} a b = a c \) implies \( ab = bac \), which is the required result for \( n = 1 \).

Assume inductively that \( a^n b = a^n c^n \). Then

\[ a^{n+1} b = a^n b a = a^n bac = ba^n c = ba^{n+1} c^{n+1} \]

since \( a \) and \( c \) commute. (Note that \( b, c \) commuting is not needed here.)

Assume inductively that \( (ab)^n = b^n a^n c^{n(n+1)/2} \) (which is true for \( n = 1 \) as noted above). Then, since \( c \) commutes with both \( a \) and \( b \),

\[
(ab)^{n+1} = (ab)^n ab = b^n a^n c^{n(n+1)/2} ab \\
= b^n a^n c^{n(n+1)/2} bac = b^n a^n b c^{n(n+1)/2} ac \\
= b^n (ba^n c^n) c^{n(n+1)/2} ac = b^{n+1} a^{n+1} c^{n(n+1)(n+2)/2}.
\]
(b) This gave trouble, because it is misprinted (but true for \( l = 1 \) nevertheless).

Note
\[
b^{-1}a^n b = (b^{-1}ab)^n = a^{kn}.
\]

So \( b^{-1}a^n b^l = a^{nk^l} \) for \( l = 1 \) and all \( n \). It is not true for \( l > 1 \).

However, assume inductively that \( b^{-1}a^n b^l = a^{nk^l} \) (true for \( l = 1 \) and all \( n \)). Then
\[
b^{-l-1}a^n b^{l+1} = b^{-1}(b^{-1}a^n b^l)b = b^{-1}a^{nk^l}b = a^{n,k^{l+1}}.
\]

M856. [1984: 283] Problems from KVANT.

(a) Construct a quadrilateral knowing the lengths of its sides and that of the segment joining the midpoints of the diagonals.

(b) Under what conditions does the problem have a solution?

Solution by Richard K. Guy, University of Calgary.

Suppose first that we are given the lengths of the sides \( 2a, 2b, 2c, 2d \), in that cyclic order, and that \( m \) is the length of the join of the midpoints of the diagonals. (Note that \( m = 0 \) implies that the quadrilateral is a parallelogram and \( a = c, b = d \) are necessary; this will be a special case of the general condition.)

(a) The construction is easily discovered from the theorem that the midpoints of the sides of any quadrilateral form a parallelogram. Draw a segment \( MN \) of length \( m \). Construct two triangles with sides \( a, c, m \) forming a parallelogram \( MANC \), say. Similarly use \( b, d, m \) to form parallelogram \( MBND \). From \( A \) draw segments \( b \) in either direction, parallel to the sides \( b \) of the parallelogram \( MBND \). From \( B \) draw segments \( a \) in either direction parallel to the sides \( a \) of the parallelogram \( MANC \). From \( C \) draw segments \( d \) in either direction parallel to the sides \( d \) of the parallelogram \( MBND \). From \( D \) draw segments \( c \) in either direction parallel to the sides \( c \) of the parallelogram \( MANC \). It is easy to prove that the 8 ends of these segments coincide in pairs at the four corners of the required quadrilateral.

(b) The construction will succeed if the parallelogram sides \( a, b, c, d \) at \( M \) (or at \( N \)) are in the required cyclic order. It may not always be possible to achieve this. The necessary and sufficient condition for success is that it is possible to arrange the sides \( 2a, 2b, 2c, 2d \) in some order so that \( a, c, m \) and \( b, d, m \) form triangles (possibly degenerate). Readers are encouraged to carry out the construction with sides \( 10, 10, 12, 12 \) and \( m = 0, 1, 2, 4, 9, 10, 11 \) and see how many different quadrilaterals (which may be nonconvex, or even crossed) can be produced in each case.

* * * * * * *
This finishes the “archive” material we have for 1984. We now turn to problems posed in the March 1989 number of the Corner.

Let $A$ be an $n \times n$ matrix such that $A^2 - 3A + 2I = 0$, where $I$ is the identity matrix and $0$ the zero matrix. Prove that $A^{2k} - (2^k + 1)A^k + 2^k I = 0$ for every natural number $k \geq 1$.

*  

Since $A^{m+n} = A^m A^n$ for all positive integers $m, n$, we see that if $f(x) = p(x)q(x)$ where $p(x), q(x)$ are polynomials in $x$, then $f(A) = p(A)q(A)$.

Let $f(x) = x^{2k} - (2^k + 1)x^k + 2^k$. Note that $f(1) = f(2) = 0$. It follows that $f(x) = (x^2 - 3x + 2)q(x)$ for some polynomial $q(x)$. Thus

$$f(A) = (A^2 - 3A + 2I)q(A) = 0q(A) = 0.$$  

Let $C$ be the set of natural numbers

$$C = \{1, 5, 9, 13, 17, 21, \ldots\}.$$  

Say that a number is *prime for $C$* if it cannot be written as a product of smaller numbers from $C$.

(a) Show that 4389 is a member of $C$ which can be represented in at least two different ways as a product of two numbers prime for $C$.

(b) Find another member of $C$ with the same property.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.
Note first that $C$ is closed under multiplication since it consists of all positive numbers of the form $4a + 1$.

(a) Since $4389 = 3 \times 7 \times 11 \times 19$ is a product of four distinct primes of the form $4b + 3$, the product of any two of these primes will have the form $4c + 1$ which clearly cannot be written as a product of smaller numbers from $C$. Hence 4389 can be written in at least three different ways as a product of two members of $C$ prime for $C$:

$$4389 = 21 \times 209 = 33 \times 133 = 57 \times 77.$$  

In fact, since it is obvious that the product of any three numbers from $3, 7, 11, 19$ is prime for $C$, there are four other ways of expressing 4389 in the described manner:

$$4389 = 3 \times 1463 = 7 \times 627 = 11 \times 399 = 19 \times 231.$$  

(b) By the argument above, to obtain another member of $C$ with the same property it suffices to replace 19 by 23, the next true prime of the form $4k + 3$ and obtain $3 \times 7 \times 11 \times 23 = 5313$.  


Given the function \( f \) defined by \( f(x) = \sqrt{4 + \sqrt{16x^2 - 8x^3 + x^4}} \).

(a) Draw the graph of the curve \( y = f(x) \).

(b) Find, without the use of integral calculus, the area of the region bounded by the straight lines \( x = 0, x = 6, y = 0 \) and by the curve \( y = f(x) \). Note: all the square roots are non-negative.

\[ \text{Solution by Seung-Jin Bang, Seoul, Republic of Korea.} \]

(a) Note that \( f(x) = \sqrt{4 + |x^2 - 4x|} \). If \( x > 4 \) or \( x < 0 \) then \( f(x) = |x - 2| \). If \( 0 < x < 4 \) then \( y = f(x) \) can be written as \( (x - 2)^2 + y^2 = (2\sqrt{3})^2 \), a circle. Thus the graph is as shown.

(b) The area of \( \triangle AOB \) and of \( \triangle BCD \) is 2, the area of sector \( ABC \) is \( \frac{1}{2} \pi (2\sqrt{3})^2 = 2\pi \), and the area of trapezoid \( CDHE \) is \( \frac{1}{2} (2 + 4) \times 2 = 6 \). Thus the area of the region is \( 8 + 2\pi \).


Let \( I_n = (n\pi - \pi/2, n\pi + \pi/2) \) and let \( f \) be the function defined by \( f(x) = \tan x - x \).

(a) Show that the equation \( f(x) = 0 \) has only one root in each interval \( I_n \), \( n = 1, 2, 3, \ldots \).

(b) If \( c_n \) is the root of \( f(x) = 0 \) in \( I_n \), find \( \lim_{n \to \infty} (c_n - n\pi) \).

\[ \text{Solution by the editors.} \]

(a) This is obvious since \( (\tan x - x)' = \frac{\sec^2 x}{1} = 1 \geq 0 \) on \( I_n \).

(b) As \( n \) goes to infinity the point of intersection of the line \( y = x \) and the graph of \( y = \tan x \) in the interval \( I_n \) goes off to infinity in each coordinate. Thus the difference \( c_n - n\pi \) must go to \( \pi/2 \).

[Editor’s note. This solution was adapted from the rather more detailed one submitted by Seung-Jin Bang, Seoul, Republic of Korea.]
of inserting $k$ bookmarks in the $16 - k$ slots (including the two “ends”) between $15 - k$ books. Therefore the total number of sequences with the described property is:

$$\sum_{k=0}^{8} \binom{16-k}{k} = 1 + 15 + 91 + 286 + 495 + 462 + 210 + 36 + 1 = 1597 < 1600.$$  

This shows that at least two students must have identical answer patterns. Indeed if there are $n$ questions the maximum possible number of students, no two with the same answer pattern, is

$$\sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k+1}{k}.$$  


Let $f$ be a continuous function on $\mathbb{R}$ such that

(i) $f(n) = 0$ for every integer $n$, and
(ii) if $f(a) = 0$ and $f(b) = 0$ then $f\left(\frac{a+b}{2}\right) = 0$, with $a \neq b$.  

Show that $f(x) = 0$ for all real $x$.


If $n$ is an integer then $f(n/2) = 0$. By induction on $m$, we have $f(n/2^m) = 0$. Let $s$ and $\epsilon$ be arbitrary real numbers. Since $f$ is continuous at $s$, there is $\delta > 0$ such that $|f(s) - f(x)| < \epsilon$ whenever $|s - x| < \delta$. Since $s$ has a 2-adic expansion, there is $n/2^m$ such that $|s - n/2^m| < \delta$. We now have $|f(s) - f(n/2^m)| = |f(s)| < \epsilon$. Hence $f(s) = 0$. (This is, of course, just the standard argument.)


Solve the following system of equations in the set of complex numbers:

$$|z_1| = |z_2| = |z_3| = 1,$$

$$z_1 + z_2 + z_3 = 1,$$

$$z_1 z_2 z_3 = 1.$$  


Let $\overline{z}$ denote the complex conjugate of $z$. We have $\overline{z_i} = 1/z_i$ for $i = 1, 2, 3$. It follows that

$$z_1 z_2 + z_2 z_3 + z_3 z_1 = \frac{1}{z_3} + \frac{1}{z_1} + \frac{1}{z_2} = \overline{z_3} + \overline{z_1} + \overline{z_2} = \overline{z_1 z_2 z_3} = 1.$$  

Consider the cubic polynomial

$$(x - z_1)(x - z_2)(x - z_3) = x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1).$$  

Since $1, \pm i$ are the roots we have that $z_1, z_2, z_3$ are equal to $1, i, -i$ in some order.

* * * * * * * *

This completes the solutions submitted for problems from the March 1989 number and this is all the space we have this month. Send me your contests!
MINI-REVIEWS
by ANDY LIU

BOOKS FROM DOVER PUBLICATIONS, INC.

The majority of Dover’s publications are reprints of excellent (otherwise, why do it?) books that are no longer available in other formats. The new editions are usually paperbound and inexpensive (averaging about $5.00 U.S. each). Often, errors in the original versions are corrected, and new material appended. While Dover has a large selection of titles in main-stream mathematics (as well as in many other areas, academic or otherwise), we will focus on the best of its line on popular mathematics. All are paperbacks.


This is one of the finest collections of problems in elementary mathematics. The 100 problems in combinatorial analysis and probability theory are all easy to understand, but some are not easy to solve, even though no advanced mathematics is required.


In this second volume, 74 problems are selected from various branches of mathematics, in particular, number theory and combinatorial geometry. This book and the earlier volume is a must for every school library.

**Mathematical Bafflers**, edited by A.F. Dunn, 1980. (337 pp.)

The bafflers in this book originally appeared as a most successful weekly corporate advertisement in technical publications. They are contributed by the readers, with a consequent diversity in their levels of sophistication. Some require almost no mathematics while others are quite demanding. However, there is a beautiful idea behind each baffler, which is compactly stated and accompanied by a cartoon.

**Second Book of Mathematical Bafflers**, edited by A.F. Dunn, 1983. (186 pp.)

This second collection of bafflers, like the earlier volume, is organized by chapters, each dealing with one area of mathematics. These include algebra, geometry, Diophantine problems and other number theory problems, logic, probability and “insight”.

**Ingenious Mathematical Problems and Methods**, by L.A. Graham, 1959. (237 pp.)

The 100 problems in this book originally appeared in the “Graham Dial”, a publication circulated among engineers and production executives. They are selected from areas not commonly included in school curricula, and have new and unusual twists that call for ingenious solutions.

**The Surprise Attack in Mathematical Problems**, by L.A. Graham, 1968. (125 pp.)

These 52 problems are selected from the “Graham Dial” on the criterion that the best solutions are not the ones the original contributors had in mind. The reader will enjoy the elegance of the unexpected approach. Like the earlier volume, the book includes a number of illustrated Mathematical Nursery Rhymes.
One Hundred Problems in Elementary Mathematics, by H. Steinhaus, 1979. (174 pp.)

The one hundred problems cover the more traditional areas of number theory, algebra, plane and solid geometry, as well as a host of practical and non-practical problems. There are also thirteen problems without solutions, some but not all of these actually having known solutions. The unsolved problems are not identified in the hope that the reader will not be discouraged from attempting them.

Fifty Challenging Problems in Probability with Solutions, by F. Mosteller, 1987. (88 pp.)

This book actually contains fifty-six problems, each with an interesting story-line. There are the familiar “gambler’s ruin” and “birthday surprises” scenarios, but with new twists. Others are unconventional, including one which turns out to be a restatement of Fermat’s Last Theorem.

Mathematical Quickies, by C. Trigg, 1985. (210 pp.)

This book contains two hundred and seventy problems. Each is chosen because there is an elegant solution. Classification by subject is deliberately avoided, nor are the problems graduated in increasing level of difficulty. This encourages the reader to explore each problem with no preconceived idea of how it should be approached.

Entertaining Mathematical Puzzles, by Martin Gardner, 1986. (112 pp.)

The master entertains with thirty-nine problems and twenty-eight quickies, covering arithmetic, geometry, topology, probability and mathematical games. There is a brief introduction to the basic ideas and techniques in each section.

Mathematical Puzzles of Sam Loyd I, edited by Martin Gardner, 1959. (167 pp.)

Sam Loyd is generally considered the greatest American puzzlist. He had a knack of posing problems in a way which attracts the public’s eye. Many of the one hundred and seventeen problems in this book had been used as novelty advertising give-aways.

Mathematical Puzzles of Sam Loyd II, edited by Martin Gardner, 1960. (177 pp.)

This book contains one hundred and sixty-six problems, most of which are accompanied by Loyd’s own illustrations, as is the case with the earlier volume. The two books represent the majority of the mathematical problems in the mammoth “Cyclopedia” by Sam Loyd, published after his death.

Amusements in Mathematics, by H.E. Dudeney, 1970. (258 pp.)

Henry Ernest Dudeney, a contemporary of Sam Loyd, is generally considered the greatest English puzzlist, and a better mathematician than Loyd. This book contains four hundred and thirty problems, representing only part of Dudeney’s output. There are plenty of illustrations in the book.

Mathematical Puzzles for Beginners and Enthusiasts, by G. Mott-Smith, 1954. (248 pp.)

This book contains one hundred and eighty-nine problems in arithmetic, logic, algebra, geometry, combinatorics, probability and mathematical games. They are both instructive and entertaining.
(418 pp.)
This is the foremost single-volume classic of popular mathematics. Written by two
distinguished mathematicians, it covers a variety of topics in great detail. After arithmeti-
cal and geometrical recreations, it moves on to polyhedra, chessboard recreations, magic
squares, map-colouring problems, unicursal problems, Kirkman’s schoolgirls problem, the
three classical geometric construction problems, calculating prodigies, cryptography and
cryptanalysis.

Mathematical Recreations, by M. Kraitchik, 1953. (330 pp.)
This is a revision of the author’s original work in French. It covers more or less the
same topics as “Mathematical Recreations and Essays”. There is a chapter on ancient and
curious problems from various sources.

The Master Book of Mathematical Recreation, by F. Schuh, 1968. (430 pp.)
This is a translation of the author’s original work in German. Four of the fifteen
chapters are devoted to the analysis of mathematical games. The remaining ones deal with
puzzles of various kinds. General hints for solving puzzles are given in the introductory
chapter. The last chapter is on puzzles in mechanics.

Puzzles and Paradoxes, by T.H. O’Beirne, 1984. (238 pp.)
Like Martin Gardner’s series, this book is an anthology of the author’s column in
New Scientist. It consists of twelve largely independent articles.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathe-
matics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Propos-
als should, whenever possible, be accompanied by a solution, references, and other insights
which are likely to be of help to the editor. An asterisk (*) after a number indicates a
problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also
be acceptable provided they are not too well known and references are given as to their
provenance. Ordinarily, if the originator of a problem can be located, it should not be
submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten
on signed, separate sheets, should preferably be mailed to the editor before August 1,
1991, although solutions received after that date will also be considered until the time when
a solution is published.

1601. Proposed by Toshio Seimiya, Kawasaki, Japan.

ABC is a right-angled triangle with the right angle at A. Let D be the foot of the
perpendicular from A to BC, and let E and F be the intersections of the bisector of \( \angle B \)
with AD and AC respectively. Prove that \( \overline{DC} > 2\overline{EF} \).
1602. Proposed by Marcin E. Kuczma, Warszawa, Poland.
Suppose \( x_1, x_2, \ldots, x_n \in [0, 1] \) and \( \sum_{i=1}^{n} x_i = m + r \) where \( m \) is an integer and \( r \in [0, 1) \). Prove that
\[
\sum_{i=1}^{n} x_i^2 \leq m + r^2.
\]

1603. Proposed by Clifford Gardner, Austin, Texas, and Jack Garfunkel, Flushing, N.Y.
Given is a sequence \( , 1, 2, \ldots \) of concentric circles of increasing and unbounded radii and a triangle \( A_1B_1C_1 \) inscribed in \( , 1 \). Rays \( A_1B_1, B_1C_1, C_1A_1 \) are extended to intersect \( , 2 \) at \( B_2, C_2, A_2 \), respectively. Similarly, \( \Delta A_2B_2C_2 \) is formed in \( , 3 \) from \( \Delta A_2B_2C_2 \), and so on. Prove that \( \Delta A_nB_nC_n \) tends to the equilateral as \( n \to \infty \), in the sense that the angles of \( \Delta A_nB_nC_n \) all tend to \( 60^\circ \).

Ever active Pythagoras recently took a stroll along a street where only Pythagoreans lived. He was happy to notice that the houses on the left side were numbered by squares of consecutive natural numbers while the houses on the right were numbered by fourth powers of consecutive natural numbers, both starting from 1. Each side had the same (reasonably large) number of houses. At some point he noticed a visitor. “It is awesome!” said the visitor on encountering Pythagoras. “Never did I see houses numbered this way.” In a short discussion that followed, the visitor heard strange things about numbers. And when it was time to part, Pythagoras asked “How many houses did you see on each side of the street?” and soon realized that counting was an art that the visitor had never mastered. “Giving answers to my questions is not my habit”, smilingly Pythagoras continued. “Go to a Crux problem solver, give the clue that the sum of the house numbers on one side is a square multiple of the corresponding sum on the other side and seek help.”

1605. Proposed by M.S. Klamkin and Andy Liu, University of Alberta.
\( ADB \) and \( AEC \) are isosceles right triangles, right-angled at \( D \) and \( E \) respectively, described outside \( \Delta ABC \). \( F \) is the midpoint of \( BC \). Prove that \( DFE \) is an isosceles right-angled triangle.

1606*. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.
For integers \( n \geq k \geq 1 \) and real \( x, 0 \leq x \leq 1 \), prove or disprove that
\[
\left( 1 - \frac{x}{k} \right)^n \geq \sum_{j=0}^{k-1} \binom{n}{j} x^j (1-x)^{n-j}.
\]
1607. Proposed by Peter Hurthig, Columbia College, Burnaby, B.C.
Find a triangle such that the length of one of its internal angle bisectors (measured from the vertex to the opposite side) equals the length of the external bisector of one of the other angles.

Suppose $n$ and $r$ are nonnegative integers such that no number of the form $n^2 + r - k(k+1), k = 1, 2, \ldots$, equals $-1$ or a positive composite number. Show that $4n^2 + 4r + 1$ is $1$, $9$, or prime.

P is a point in the interior of a tetrahedron $ABCD$ of volume $V$, and $F_a, F_b, F_c, F_d$ are the areas of the faces opposite vertices $A, B, C, D$, respectively. Prove that

$$PA \cdot F_a + PB \cdot F_b + PC \cdot F_c + PD \cdot F_d \geq 9V.$$ 

Consider the multiplication $d \times dd \times ddd$, where $d < b - 1$ is a nonzero digit in base $b$, and the product (base $b$) has six digits, all less than $b - 1$ as well. Suppose that, when $d$ and the digits of the product are all increased by $1$, the multiplication is still true. Find the lowest base $b$ in which this can happen.

** SOLUTIONS **

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


Let $m_a, m_b, m_c$ denote the lengths of the medians to sides $a, b, c$, respectively, of triangle $ABC$, and let $M_a, M_b, M_c$ denote the lengths of these medians extended to the circumcircle of the triangle. Prove that

$$\frac{M_a}{m_a} + \frac{M_b}{m_b} + \frac{M_c}{m_c} \geq 4.$$ 

IV. Generalization by Dragoljub M. Milošević, Pranjani, Yugoslavia.
In his solution [1982: 308–309], M.S. Klamkin showed that the problem is equivalent to

$$\sum \frac{a^2}{2(b^2 + c^2) - a^2} \geq 1$$

where the sum is cyclic over $a, b, c$. He also proved the related result

$$\sum \frac{a}{k(b + c) - a} \geq \frac{3}{2k - 1}$$
where \( k \geq 1 \), and suggested the more general problem of finding all \( k \geq 2 \) such that
\[
\sum a^2 \frac{a^2}{k(b^2 + c^2) - a^2} \geq \frac{3}{2k - 1},
\]
or even all \( k \) and \( n \) such that
\[
\sum a^n \frac{a^n}{k(b^n + c^n) - a^n} \geq \frac{3}{2k - 1}.
\]
Here we prove the inequality
\[
\sum a^\lambda \frac{a^\lambda}{k(b^n + c^n) - a^n} \geq \frac{3^{2-\lambda}(a^n + b^n + c^n)^{\lambda-1}}{2k - 1}
\]
for \( \lambda = 1 \) or \( \lambda \geq 2 \), and for \( k \geq 2^{n-1} \) where \( n \geq 1 \), generalizing the above inequalities.

Start with the function
\[
f(x) = \frac{x^\lambda}{p - qx}, \quad 0 < x < p/q,
\]
where \( p > 0, q > 0, \lambda \in \{1\} \cup [2, \infty) \). Since
\[
f''(x) = \frac{x^{\lambda-2}}{(p - qx)^3}[(\lambda - 1)(\lambda - 2)q^2x^2 - 2pq\lambda(\lambda - 2)x + p^2\lambda(\lambda - 1)] > 0
\]
for \( \lambda \geq 2 \) and also for \( \lambda = 1 \), function \( f \) is convex, so for \( 0 < x_i < p/q \)
\[
\sum_{i=1}^{3} \frac{x_i^\lambda}{p - qx_i} = \sum_{i=1}^{3} f(x_i) \geq 3f \left( \frac{x_1 + x_2 + x_3}{3} \right) = \frac{3^{2-\lambda}(x_1 + x_2 + x_3)^{\lambda}}{3p - q(x_1 + x_2 + x_3)}.
\]
Putting in (2)
\[
p = (a^n + b^n + c^n)k \quad , \quad q = k + 1 \quad , \quad x_1 = a^n, \ x_2 = b^n, \ x_3 = c^n,
\]
we obtain the desired inequality (1). Note that, for example, \( x_1 < p/q \) is equivalent to
\[
a^n < k(b^n + c^n),
\]
so since \( k \geq 2^{n-1} \) it is enough to prove that
\[
(b + c)^n < 2^{n-1}(b^n + c^n),
\]
which is true by the convexity of \( x^n \) for \( n \geq 1 \).

\( \triangle ABC \) is acute angled with sides \( a, b, c \) and has circumcircle \( \odot \), with centre \( O \). The inner bisector of \( \angle A \) intersects \( \odot \), for the second time in \( A_1 \). \( D \) is the projection on \( AB \) of \( A_1 \). \( L \) and \( M \) are the midpoints of \( CA \) and \( AB \) respectively. Show that

(i) \( AD = \frac{1}{2}(b + c) \);
(ii) \( A_1D = OM + OL \).

II. Comment by Toshio Seimiya, Kawasaki, Japan.

[This is in response to a question of the editor [1990:93].]

Part (i) remains true for nonacute triangles. Here is a proof for all triangles. Letting \( E \) be the foot of the perpendicular from \( A_1 \) to \( AC \), we get \( AD = AE \) and \( A_1D = A_1E \). Because \( BA_1 = CA_1 \), we then have \( \triangle A_1DB \equiv \triangle A_1EC \), and therefore \( BD = CE \). Thus

\[ b + c = AB + AC = AD + AE = 2AD, \]

and (i) follows.

Part (ii) does not hold for all triangles. If \( \angle B > 90^\circ \) we get \( A_1D = OM - OL \), and if \( \angle C > 90^\circ \) we get \( A_1D = OL - OM \).

By using the relation (ii) (for acute triangles) we have an alternate proof of the well known theorem: \textit{if \( \triangle ABC \) is an acute triangle with circumcentre \( O \), circumradius \( R \), and inradius \( r \), and \( L, M, N \) are the midpoints of the sides, then}

\[ OL + OM + ON = R + r. \]

(N.A. Court, College Geometry, p. 73, Thm. 114). [Editor’s note: this theorem was also used by Seimiya in his proof of Crux 1488, this issue.]

Let \( I \) be the incen ter of \( \triangle ABC \), and let \( T, S \) be the feet of the perpendiculars from \( I \) to \( AD, A_1D \) respectively. Then we get \( SD = IT = r \).

It is well known that \( A_1I = A_1C \), and because

\[ \angle A_1IS = \angle A_1AB = \angle A_1CB \]

and

\[ \angle ISA_1 = \angle CAN = 90^\circ, \]

we get \( \triangle A_1IS \equiv \triangle A_1CN \), from which we have \( A_1S = A_1N \). Using (ii) we have

\[ OL + OM + ON = A_1D + ON = A_1S + SD + ON = A_1N + r + ON = R + r. \]
Another solution was received from K.R.S. SASTRY, Addis Ababa, Ethiopia, in which he also shows that converses of (i) and (ii) need not hold, and finds similar formulae in the case that $AA_1$ is the external bisector of $\angle A$.

\[
\begin{align*}
* & \quad * & \quad * & \quad * & \quad * & \quad *
\end{align*}
\]


If the Nagel point of a triangle lies on the incircle, prove that the sum of two of the sides of the triangle equals three times the third side.

IV. Comment and solution by Dan Sokolowsky, Williamsburg, Virginia.

This is in response to L.J. Hut's claim [1990: 182] that, if $D$ is the point at which the incircle of $\triangle ABC$ touches $AB$, then the incircles of triangles $ABC$ and $A'B'C'$ touch at $D$. The claim is true provided that some additional assumption (such as (2) below) is made.

First, it is easy to see (referring to the figure on [1990: 181]) that this claim is equivalent to

\[NI \perp AB\text{ at } D. \tag{1}\]

We show, assuming that the Nagel point of $\triangle ABC$ lies on its incircle, that (1) is equivalent to: $AB$ is the shortest side of $\triangle ABC$, that is, to

\[c \leq a, b \tag{2}\]

(which, incidentally, would justify Hut's selection of $AB$ over the other edges of $\triangle ABC$).

In the figure, let $w$ denote the incircle of $\triangle ABC$, touching $AB$ at $D$, $CA$ at $Z$, $CB$ at $Y$. Let $P, Q$ be points on $CA, CB$ respectively such that $PQ \parallel AB$ and $PQ$ touches $w$, say at $F$. Obviously $FI \perp PQ$, hence $FI \perp AB$. Thus to show (2) implies (1) it will suffice to show that (2) implies $F$ is the Nagel point $N$ of $\triangle ABC$.

Let $CF$ meet $w$ again at $F'$ and $AB$ at $V$. Note that $w$ is the excircle of $\triangle CPQ$ on side $PQ$, hence the Nagel point of $\triangle CPQ$ lies on $CF$. Then, since $\triangle ABC \sim \triangle CPQ$, the Nagel point $N$ of $\triangle ABC$ lies on $CV$. By hypothesis, it also lies on $w$, hence it is either $F$ or $F'$. We can assume that $F'$ lies on the arc $DZ$. Then $BF'$ meets $AZ$ at a point $X$. If $F'$ were the Nagel point of $\triangle ABC$ we would then have

\[s - c = AX < AZ = s - a\]

($s$ the semiperimeter), which implies $a < c$, contradicting (2). It follows that $F'$ cannot be the Nagel point of $\triangle ABC$, which must then be $F$, so (1) follows. Conversely, if (1) holds,
then by the preceding argument $NI$ is perpendicular to the shortest side of $\triangle ABC$, which must therefore be $AB$, so (2) holds.

A simple proof of the problem could now go as follows. Parts marked in the adjoining figure have the same meaning as before, but we assume (2), so that by the above the point marked $N$ is the Nagel point of $\triangle ABC$. Let $AN$ meet $BC$ at $T$, and let $s'$ be the semiperimeter of $\triangle CPQ$. We then have

$$2s' = (CP + PN) + (NQ + QC) = CZ + CY = 2(s - c).$$

Since $\triangle CPQ \sim \triangle C AB$,

$$\frac{CN}{CV} = \frac{s'}{s} = \frac{s - c}{s}$$

and hence

$$\frac{CN}{NV} = \frac{s - c}{c}.$$

Since $N$ is the Nagel point of $\triangle ABC$,

$$AV = s - b, \ BT = s - c, \ TC = s - b.$$ By Menelaus applied to $\triangle CVB$,

$$\frac{1}{\frac{CN}{NV} \cdot \frac{AV}{AB} \cdot \frac{BT}{TC}} = \frac{s - c}{c} \cdot \frac{s - b}{c} \cdot \frac{s - c}{s - b} = \frac{(s - c)^2}{c^2}.$$ Thus $s - c = c$, which implies $a + b = 3c$.

Sokolowsky also pointed out a typo on [1990: 182], line 2: $DG = 2GD'$ should read $2DG = GD'$.

L.J. Hut submitted a further clarification of his solution, showing not only that $IN \perp AB$ implies $a + b = 3c$, but also that $IN \parallel AB$ implies $a + b = 2c$. Interesting! Any comments?

* * * * * *


Let $A, B, C$ be points on a fixed circle with $B, C$ fixed and $A$ variable. Points $D$ and $E$ are on segments $BA$ and $CA$, respectively, so that $BD = m$ and $CE = n$ where $m$ and $n$ are constants. Points $P$ and $Q$ are on $BC$ and $DE$, respectively, so that

$$BP : PC = DQ : QE = k,$$
also a constant. Prove that the length of \(PQ\) is a constant. (This is not a new problem. A reference will be given when the solution is published.)

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

Draw the lines \(XB\) and \(YP\) parallel to \(CE\) and of the same length, so that \(X, Y, E\) are collinear and \(XE \parallel BC\). Also find \(Q\) on \(DE\) so that \(YQ \parallel XD\). Then

\[
\frac{DQ}{QE} = \frac{XY}{YE} = \frac{BP}{PC} = k, \tag{1}
\]

so \(Q\) is as defined in the problem. We get the fixed triangle \(BDX\) with sides \(BD = m\) and \(BX = CE = n\) and included angle \(\alpha\) (since \(A\) lies on a fixed circle). Thus for every position of \(A\) the side \(XD\) is fixed. From (1) the side \(QY\) is fixed. Since also \(PY = CE = n\) and \(\angle PYQ = \angle BXD\) is fixed, side \(PQ\) is constant.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; JORDI DOU, Barcelona, Spain; C. FESTREETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposers.

Although nobody mentioned it, it appears that point \(A\) must be restricted to one of the arcs \(BC\) of the circle.

The problem was found by the proposers in Journal de Mathématiques Élémentaires (1912).

* * * * *

1482. Proposed by M.S. Klamkin, University of Alberta.

If \(A, B, C\) are vectors such that

\[|A| = |B| = |C| = |A + B + C|,\]

prove that

\[|B \times C| = |A \times (B + C)|.\]
I. Solution by Hans Lausch, Monash University, Melbourne, Australia.

Let \( W, X, Y, Z \) be points in \( \mathbb{R}^3 \) such that
\[
\overrightarrow{ZX} = \mathbf{A}, \quad \overrightarrow{XW} = \mathbf{B}, \quad \text{and} \quad \overrightarrow{WY} = \mathbf{C}.
\]
Then
\[
\overrightarrow{ZX} = \overrightarrow{ZY} = \overrightarrow{WX} = \overrightarrow{WY},
\]
so
\[
\Delta XYZ \cong \Delta XYZ.
\]

Therefore
\[
|\mathbf{B} \times \mathbf{C}| = 2 \cdot \text{area}(XYW) = 2 \cdot \text{area}(XYZ) = |\mathbf{A} \times (\mathbf{B} + \mathbf{C})|.
\]


It suffices to prove the result if
\[
|\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}| = |\mathbf{A} + \mathbf{B} + \mathbf{C}| = 1.
\]
Then
\[
1 = |\mathbf{A} + \mathbf{B} + \mathbf{C}|^2 = \mathbf{A}^2 + 2\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) + \mathbf{B}^2 + 2\mathbf{B} \cdot \mathbf{C} + \mathbf{C}^2
= 3 + 2\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) + 2\mathbf{B} \cdot \mathbf{C},
\]
so
\[
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = -(\mathbf{B} \cdot \mathbf{C} + 1).
\]
It follows from
\[
(u \times v)^2 = u^2v^2 - (u \cdot v)^2
\]
(twice) that
\[
|\mathbf{A} \times (\mathbf{B} + \mathbf{C})|^2 = \mathbf{A}^2(\mathbf{B} + \mathbf{C})^2 - (\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}))^2
= (\mathbf{B} + \mathbf{C})^2 - (\mathbf{B} \cdot \mathbf{C} + 1)^2
= 1 - (\mathbf{B} \cdot \mathbf{C})^2 = |\mathbf{B} \times \mathbf{C}|^2,
\]
which completes the proof.

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; JORDI DOU, Barcelona, Spain; G.P. HENDERSON, Campbelicroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WAL-THER JANOUS, Ursulengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; CHRIS WILDHAGEN, Breda, The Netherlands; and the proposer.

The solutions of Dou, Henderson, and McCallum were similar to solution I.

The proposer gave the following geometric interpretation. Consider the transformation
\[
\mathbf{X} = \frac{\mathbf{B} \times \mathbf{C}}{[ABC]}, \quad \mathbf{Y} = \frac{\mathbf{C} \times \mathbf{A}}{[ABC]}, \quad \mathbf{Z} = \frac{\mathbf{A} \times \mathbf{B}}{[ABC]},
\]
where \([ABC] = A \cdot (B \times C)\). Then it is known (e.g., Spiegel, Vector Analysis, Schaum, 1959, Chapter 2, exercises 53(c) and 103) that, reciprocally,

\[
A = \frac{Y \times Z}{[XYZ]} , \quad B = \frac{Z \times X}{[XYZ]} , \quad C = \frac{X \times Y}{[XYZ]} ,
\]

and

\([XYZ] \cdot [ABC] = 1\).

Substituting for \(A, B, C\) in the original problem, one gets the dual problem: if \(X, Y, Z\) are vectors such that

\[
|Y \times Z| = |Z \times X| = |X \times Y| = |(Y \times Z) + (Z \times X) + (X \times Y)|.
\]

then (by symmetry)

\[
|X| = |Y - Z| , \quad |Y| = |Z - X| , \quad |Z| = |X - Y| .
\]

Now consider a tetrahedron \(PXYZ\) where \(\overline{PX} = X, \overline{PY} = Y, \overline{PZ} = Z\). Then the above shows that if the four faces of a tetrahedron have equal areas, the tetrahedron must be isosceles, i.e., opposite pairs of edges are congruent. For a geometric proof, see N. Altshiller Court, Modern Pure Solid Geometry, Macmillan, N.Y., 1935, Corollary 307.

\[ * \quad * \quad * \quad * \quad * \]

**1483.** Proposed by George Tsintsifas, Thessaloniki, Greece.

Let \(A'B'C'\) be a triangle inscribed in a triangle \(ABC\), so that \(A' \in BC, B' \in CA, C' \in AB\), and so that \(A'B'C'\) and \(ABC\) are directly similar.

(a) Show that, if the centroids \(G, G'\) of the triangles coincide, then either the triangles are equilateral or \(A', B', C'\) are the midpoints of the sides of \(\triangle ABC\).

(b) Show that if either the circumcenters \(O, O'\) or the incenters \(I, I'\) of the triangles coincide, then the triangles are equilateral.

**Solution by Jordi Dou, Barcelona, Spain.**

When \(ABC\) is equilateral the triangles \(A'B'C'\) are concentric with \(ABC\) and all of \(G', O', I'\) coincide with \(O = G = I\). Hereafter we suppose that \(ABC\) is not equilateral.

Let \(T_0'\) be the triangle whose vertices are the midpoints \(A'_0, B'_0, C'_0\) of \(BC, CA, AB\), respectively. The perpendicular bisectors of the sides of \(ABC\) (through \(A'_0, B'_0, C'_0\)) concur at the circumcentre \(O\) of \(ABC\). Any triangle \(T' = A'B'C'\) as described in the problem can be obtained by applying a rotation of centre \(O\) and angle \(x\), followed by a homothety of centre \(O\) and ratio \(r = 1/\cos x\), to triangle \(T'_0\). \[Editor's note. Can someone supply a reference?\] The vertices \(A'_x, B'_x, C'_x\) of the resulting triangle \(T'_x\) will be on the sides of \(ABC\). Since \(T'_0\) is similar to \(ABC\), \(T'_x\) will also be similar to \(ABC\) for every \(x\). Note that \(O\) is the orthocentre \(H'_0\) of \(T'_0\) and thus also the orthocentre \(H'_x\) of every \(T'_x\). Let \(G'_x\) be the centroid of \(T'_x\). From \(G'_xO = (1/\cos x)G'_0O\) and \(\angle G'_0OG'_x = x\) it follows that \(G'_xG'_0 \perp G'_0O\). Thus the locus of the centroids \(G'_x\) is the line through \(G'_0\) perpendicular to \(OG'_0\). Analogously
the locus of the circumcentres $O'_x$ and of the incentres $I'_x$ of the triangles $T'_x$ are the lines through $O'_0$ and $I'_0$ perpendicular to $OO'_0$ and $OI'_0$ respectively.

Since $G = G'_0$, $G = G'_x$ only for $x = 0$; this solves (a). Since $O = H'_0 \neq O'_0$ unless $T'_0$ (i.e., $ABC$) is equilateral, $O$ cannot coincide with $O'_0$; this solves the first part of (b). Finally, if $I = I'_x$ then, since $G = G'_0$ is on the line $II'_0$, $\angle HIG = \angle H_0I'_0G'_0 = 90^\circ$.

Editor's note. Dou claimed not to have finished the proof of the impossibility of $I = I'_x$, but it now follows immediately from the known result that $GH^2 \geq GI^2 + IH^2$, with equality only for the equilateral triangle. See p. 288 of Mitrović, Pečarić, and Volenec, *Recent Advances in Geometric Inequalities*, or the solution of *Crux* 260 [1978: 58].

Also solved by the proposer. Part (a) only was solved by JILL HOUGHTON, Sydney, Australia.

For part (a), the proposer simply applied his two earlier problems *Crux* 1464 [1990: 282] and *Crux* 1455 [1990: 249].

* * * * *

1484. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $0 < r, s, t \leq 1$ be fixed. Show that the relation

$$r \cot rA = s \cot sB = t \cot tC$$

holds for exactly one triangle $ABC$, and that this triangle maximizes the expression

$$\sin rA \sin sB \sin tC$$

over all triangles $ABC$.

Solution by Murray S. Klamkin, University of Alberta.

At most one of the angles $rA, sB, tC$ can be greater than $\pi/2$ so that, by the equality conditions, none of them are greater than $\pi/2$. Consequently the given cotangents are monotonic in their angle argument. Now assume that $A, B, C$ and $A', B', C'$ are different solutions. Since $A' + B' + C' = A + B + C = \pi$, one pair of angles from $A', B', C'$ must be bigger and smaller than the corresponding pair in $A, B, C$. This gives a contradiction since the cotangents are monotonic. Consequently $ABC$ is unique.

To maximize $\sin rA \sin sB \sin tC$ we take logs and use Lagrange multipliers with Lagrangian

$$\mathcal{L} = \log \sin rA + \log \sin sB + \log \sin tC - \lambda (A + B + C).$$

Then

$$\frac{\partial \mathcal{L}}{\partial A} = \frac{\partial \mathcal{L}}{\partial B} = \frac{\partial \mathcal{L}}{\partial C} = 0$$

yields the given cotangent relations for the maximum. On the boundary, i.e. for $A$ or $B$ or $C$ equal to 0, we obtain the minimum value 0.

More generally, a similar argument goes through to maximize

$$\sin^\alpha rA \sin^\beta sB \sin^\gamma tC$$
with the additional condition \( u, v, w > 0 \). Here the maximizing equations are

\[
ur \cot rA = vs \cot sB = wt \cot tC.
\]

Also, a similar argument for maximizing

\[
\cos u rA \cos v sB \cos w tC
\]

does not go through the same way. Here the extremal equations are

\[
ur \tan rA = vs \tan sB = wt \tan tC.
\]

However, we now have to check the boundary. This entails setting one and then two of \( A, B, C \) equal to 0. Then we have to decide the absolute minimum and maximum from these seven possibilities. We leave this as an open problem.

Also solved by the proposer, who mentions that the problem contains as special cases the items 2.10-2.13 of Bottema et al, Geometric Inequalities.

By “exactly one” triangle (the editor’s wording) was meant of course “up to similarity”!

* * * * *


From a deck of 52 cards, 13 are chosen. Replace one of them by one of the remaining 39 cards. Continue the process until the initial set of 13 cards reappears. Is it possible that all the \( \binom{52}{13} \) combinations appear on the way, each exactly once?

Comment by Stanley Rabinowitz, Westford, Massachusetts.

It is known that the subsets of size \( k \) from a set of size \( n \) can be arranged in a circular list such that adjacent sets in the list differ by the replacement of one element by another. (The subsets are said to be in revolving door order.) A reference is [1] in which an algorithm for forming such a list is given (not just an existence proof).

The idea behind the algorithm is as follows. If \( A(m, l) \) denotes a list of all the \( l \)-subsets of \( \{1, 2, \ldots, m\} \) arranged in revolving door order beginning with \( \{1, 2, \ldots, l\} \) and ending with \( \{1, 2, \ldots, l-1, m\} \), then it can be shown that

\[
A(n, k) = A(n - 1, k) \quad \overline{A(n - 1, k - 1)} \times \{n\},
\]

where the bar means that the order of the list is reversed and the cross means that the element \( n \) is appended to each subset in the list. It is easy to check that if \( A(n - 1, k) \) and \( A(n - 1, k - 1) \) are in revolving door order, then so is \( A(n, k) \). It follows by induction that the list \( A(n, k) \) exists for each \( n \) and \( k \).

Reference:

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and the proposer.
**1487. Proposed by Kee-Wai Lau, Hong Kong.**

Prove the inequality

$$x + \sin x \geq 2 \log(1 + x)$$

for $x > -1$.

*Combined solutions of Richard I. Hess, Rancho Palos Verdes, California, and the proposer.*

Let

$$f(x) = x + \sin x - 2 \log(1 + x) \quad , \quad x > -1.$$ 

We have

$$x + \sin x = 2x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = 2x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

and

$$2 \log(1 + x) = 2x - \frac{2x^2}{2} + \frac{2x^3}{3} - \frac{2x^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1} x^k}{k},$$

and thus

$$f(x) \quad = \quad x^2 \left(1 - \frac{2x}{3} + \frac{x^3}{3!}\right) + x^4 \left(1 - \frac{4x}{5} + \frac{2x^2}{5!}\right) + x^6 \left(1 - \frac{6x}{7} + \frac{3x^3}{7!}\right) + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \left(1 - \frac{2k}{2k+1}x + \frac{(-1)^k x}{(2k+1)!}x\right).$$

**Case (i):** $-1 < x \leq 1$. Then for each $k \geq 1$,

$$1 - \frac{2k}{2k+1}x + \frac{(-1)^k x}{(2k+1)!}x > 1 - \frac{2k}{2k+1} - \frac{k}{(2k+1)!} = \frac{(2k)! - k}{(2k+1)!} > 0,$$

so $f(x) \geq 0$.

**Case (ii):** $x \geq 4.5$. Since $x - 2 \log(1 + x)$ increases for $x > 1$, and

$$f(4.5) \geq 4.5 - 2 \log 5.5 \approx 0.0905038,$$

$f(x) > 0$ for $x \geq 4.5$.

**Case (iii):** $1 < x < 4.5$. The functions

$$g(x) = x + \sin x \quad \text{and} \quad h(x) = 2 \log(1 + x)$$

are both nondecreasing, and by means of a calculator we check that

$$g(a) - h(a + 0.05) > 0$$

for $a = 1, 1.05, 1.1, 1.15, \ldots, 4.5$. [Editor’s note. He’s right! In fact the smallest value for $g(a) - h(a + 0.05)$ you get is $g(4.05) - h(4.1) \approx 0.0029937$.] Hence for $x \in (a, a + 0.05]$ where $a$ is any of the above values,

$$f(x) = g(x) - h(x) \geq g(a) - h(a + 0.05) > 0.$$
Thus \( f(x) > 0 \) on \((1, 4.5)\) as well.

Also solved by WALther Janous, Ursulengymnasium, Innsbruck, Austria.
There were two incorrect solutions sent in.

Equality holds at \( x = 0 \), and nearly does again at around \( x = 4 \); both Hess and Janous found a relative minimum for the function \( f \) at \( x \approx 4.06268, f(x) \approx 0.022628 \). All three correct proofs had difficulty getting past this point.

* * * * *

1488. Proposed by Avinoam Freedman, Teaneck, New Jersey.

Prove that in any acute triangle, the sum of the circumradius and the inradius is less than the length of the second-longest side.

I. Solution by Toshi Seimiya, Kawasaki, Japan.

In the figure, \( O \) is the circumcenter of a triangle \( ABC \) with circumradius \( R \) and inradius \( r \), and \( OL, OM, ON \) are the perpendiculars to \( BC, CA, AB \), respectively. Then we have

\[ R + r = OL + OM + ON \]

(Court, College Geometry, p. 73, Theorem 114).
We put \( BC = a, CA = b, AB = c \), and assume without loss of generality that \( a \geq b \geq c \). Let \( CH \) be the altitude and \( S \) the area of \( \triangle ABC \). Then

\[ c \cdot CH = 2S = a \cdot OL + b \cdot OM + c \cdot ON \]

\[ \geq c(OL + OM + ON), \]

so we have (since \( \angle A \) is acute)

\[ b > CH \geq OL + OM + ON = R + r. \]

II. Solution by the proposer.

Let the triangle be \( ABC \) with \( a \leq b \leq c \). Since \( 90^\circ > B > 90^\circ - A \), we have \( \cos B < \cos(90^\circ - A) = \sin A \), and similarly \( \cos A < \sin B \). Therefore

\[ (1 - \cos A)(1 - \cos B) > (1 - \sin A)(1 - \sin B). \]

Expanding and rearranging, we find

\[ \sin A + \sin B > \cos A + \cos B + (\sin A \sin B - \cos A \cos B) \]

\[ = \cos A + \cos B + \cos C. \]

Since

\[ \sin A = \frac{a}{2R}, \text{ etc.}, \quad \text{and} \quad \sum \cos A = 1 + \frac{r}{R} = \frac{R + r}{R}, \]
where $R$ is the circumradius and $r$ the inradius, we see that

$$R + r < \frac{a + b}{2} \leq b.$$  

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; DUANE M. BROLINE, Eastern Illinois University, Charleston; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and BOB PRIELIPP, University of Wisconsin—Oshkosh. Two incorrect solutions were sent in.

The solutions of Bellot Rosado and of Priellup were similar to but shorter than Solution I, appealing to item 11.16 of Bottema et al, Geometric Inequalities for the inequality $\overline{CH} \geq R + r$.

* * * * *

1489. Proposed by M. Selby, University of Windsor.

Let

$$A_n = (7 + 4\sqrt{3})^n,$$

where $n$ is a positive integer. Find a simple expression for $1 + [A_n] - A_n$, where $[x]$ is the greatest integer less than or equal to $x$.

Solution by Guo-Gang Gao, student, Université de Montréal.

If

$$(2 + \sqrt{3})^{2n} = a_n + b_n\sqrt{3},$$

where $a_n$ and $b_n$ are integers, then

$$(2 - \sqrt{3})^{2n} = a_n - b_n\sqrt{3},$$

which results from the binomial formula. Therefore

$$(2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n}$$

is an integer. Since $(2 - \sqrt{3})^{2n} < 1$, it follows that

$$[2 + \sqrt{3}]^{2n} = (2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n} - 1.$$

The above equation can be rewritten as

$$[(7 + 4\sqrt{3})^n] = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n - 1.$$  

Therefore,

$$1 + [A_n] - A_n = 1 + [(7 + 4\sqrt{3})^n] - (7 + 4\sqrt{3})^n = (2 - \sqrt{3})^{2n}.$$  

Also solved by HAYO AHLBURG, Benidorm, Spain; CURTIS COOPER, Central Missouri State University; NICOS D. DIAMANTIS, student, University of Patras, Greece;
Several solvers gave generalizations.

\[ x + y + z + z \leq k \sqrt{x + y + z} \]

for all positive \( x, y, z \).

\[ x + y = c^2, \quad y + z = a^2, \quad z + x = b^2, \]

where \( a, b, c > 0 \) and we assume \( a \geq b, c \). Solving these,

\[ x = \frac{-a^2 + b^2 + c^2}{2}, \quad y = \frac{a^2 - b^2 + c^2}{2}, \quad z = \frac{a^2 + b^2 - c^2}{2}, \]

and the inequality becomes

\[ \frac{-a^2 + b^2 + c^2}{c} + \frac{a^2 - b^2 + c^2}{a} + \frac{a^2 + b^2 - c^2}{b} \leq \frac{5}{2\sqrt{2}} \sqrt{a^2 + b^2 + c^2}. \]

It is easy to see that

\[ \frac{a + \sqrt{b^2 + c^2}}{\sqrt{2}} \leq \sqrt{a^2 + b^2 + c^2} \]

[put \( \sqrt{b^2 + c^2} = d \) and square both sides], so we can replace the right side of (1) by

\[ \frac{5}{4}(a + \sqrt{b^2 + c^2}). \]
The left side can be written

\[(a + b + c) + \frac{(a + b + c)(a - b)(a - c)(c - b)}{abc} \cdot \]

Since the second part of this just changes sign if we interchange \(b\) and \(c\), while the rest of the inequality does not change, we can assume that this term is positive, i.e., \(c \geq b\). Multiplying by \(4abc\), the inequality to be proved becomes

\[4abc(a + b + c) + 4(a + b + c)(a - b)(a - c)(c - b) \leq 5abc(a + \sqrt{b^2 + c^2}),\]

or \(f(a) \leq 0\), where

\[f(a) = 4a^3(c - b) - a^2bc + a(4b^3 + 4b^2c + 4bc^2 - 4c^3 - 5bc\sqrt{b^2 + c^2}) + 4bc(c^2 - b^2).\]

Since \(x \geq 0\), \(a \leq \sqrt{b^2 + c^2}\) and we are to show that \(f(a)\) is negative for

\[b \leq c \leq a \leq \sqrt{b^2 + c^2}.\]  \hspace{1cm} (2)

If \(b = c\),

\[f(a) = -ab^2[(a - b) + (5\sqrt{2} - 7)b] < 0.\]

If \(b < c\), \(f(a)\) is a cubic with first and last coefficients greater than 0. We have

\[f(-\infty) < 0, \ f(0) > 0, \ f(\infty) > 0,\]

and we find

\[f(c) = -bc^2(5\sqrt{b^2 + c^2} - 4b - 3c) < 0\]

because

\[25(b^2 + c^2) - (4b + 3c)^2 = (3b - 4c)^2 > 0;\]

further,

\[f(\sqrt{b^2 + c^2}) = 2bc(4b\sqrt{b^2 + c^2} - 5b^2 - c^2)\]

\[= -2bc(\sqrt{b^2 + c^2} - 2b)^2 \leq 0.\]

We see that \(f\) has three real zeros. One is negative, one is between 0 and \(c\), and one is equal to or greater than \(\sqrt{b^2 + c^2}\). Therefore \(f\) does not change sign in \(c \leq a \leq \sqrt{b^2 + c^2}\) and is negative for the whole interval, except possibly at \(\sqrt{b^2 + c^2}\).

The only solution of \(f(a) = 0\) that satisfies (2) is \(a = \sqrt{b^2 + c^2}\), and then only if \(\sqrt{b^2 + c^2} = 2b\); that is, when \((a, b, c)\) are proportional to \((2, 1, \sqrt{3})\) and so \((x, y, z)\) are proportional to \((0, 3, 1)\).

Also solved (obtaining the same value \(k = 5/4\) via a somewhat longer argument) by MARCIN E. KUCZMA, Warszawa, Poland.

At the end of his proof, Kuczma uses the same substitution

\[y + z = a^2, \ z + x = b^2, \ x + y = c^2\]
as Henderson to rewrite the inequality in the form (1). He then observes that since the numerators on the left side of (1) are all positive, $a, b, c$ are the sides of an acute triangle $ABC$. Letting $AP, BQ, CR$ be the altitudes, he obtains

$$\frac{a^2 + b^2 - c^2}{2b} = \frac{2abc \cos C}{2b} = a \cos C = CQ,$$

and thus the inequality takes the form

$$AR + BP + CQ \leq \frac{5}{4\sqrt{2}} \sqrt{a^2 + b^2 + c^2},$$

with equality for the $30^\circ - 60^\circ - 90^\circ$ triangle. (Not exactly acute, as Kuczma notes!)

A lovely problem! It seems almost ungrateful to ask if there is a generalization to $n$ variables . . . .

* * * * *


In triangle $ABC$, the internal bisector of $\angle A$ meets $BC$ at $D$, and the external bisectors of $\angle B$ and $\angle C$ meet $AC$ and $AB$ (produced) at $E$ and $F$ respectively. Suppose that the normals to $BC, AC, AB$ at $D, E, F$ respectively, meet. Prove that $AB = AC$.

I. Solution by Toshio Seimiya, Kawasaki, Japan.

Because the normals to $BC, AC, AB$ at $D, E, F$ are concurrent (at $Q$, say), we have

$$\overline{BD}^2 + \overline{DQ}^2 = \overline{BF}^2 + \overline{FQ}^2,$$

$$\overline{CE}^2 + \overline{EQ}^2 = \overline{CD}^2 + \overline{DQ}^2,$$

$$\overline{AF}^2 + \overline{FQ}^2 = \overline{AE}^2 + \overline{EQ}^2,$$

so, adding,

$$(\overline{BD}^2 - \overline{DC}^2) + (\overline{CE}^2 - \overline{EA}^2) + (\overline{AF}^2 - \overline{FB}^2) = 0. \quad (1)$$

We put $\overline{BC} = a, \overline{CA} = b, \overline{AB} = c$. In the figure we are assuming $a < b, c$. As $AD$ is the bisector of $\angle A$, we get $\overline{BD} : \overline{DC} = c : b$, so we have

$$\overline{BD} = \frac{ac}{b + c}, \quad \overline{DC} = \frac{ab}{b + c}.$$

Similarly $\overline{AE} : \overline{EC} = c : a$ and $\overline{AF} : \overline{FB} = b : a$, so we get

$$\overline{CE} = \frac{ab}{c - a}, \quad \overline{EA} = \frac{bc}{c - a},$$

$$\overline{BF} = \frac{ac}{b - a}, \quad \overline{AF} = \frac{bc}{b - a}.$$
Therefore from (1) we get
\[
\frac{a^2(c^2 - b^2)}{(b + c)^2} + \frac{b^2(a^2 - c^2)}{(c - a)^2} + \frac{c^2(b^2 - a^2)}{(b - a)^2} = 0,
\]
or
\[
\frac{a^2(c - b)}{b + c} - \frac{b^2(a + c)}{c - a} + \frac{c^2(b + a)}{b - a} = 0. \tag{2}
\]
In the case \(a > b, c\), or \(b > a > c\), or \(b < a < c\), we have (2) similarly. [Because the above formulas for \(CE\), etc. will change only in sign.—Ed.] The left side of (2) becomes
\[
\frac{(c - b)(a + b)(a + c)(b + c - a)^2}{(b + c)(c - a)(b - a)}.
\]

[Editor’s note. Seimiy gave an algebraic derivation. Can anyone find a slick reason why the left side of (2) factors so conveniently?] Because \(a, b, c > 0\) and \(b + c > a\), we obtain \(c - b = 0\). This implies \(AB = AC\).

II. Solution by R.H. Eddy, Memorial University of Newfoundland.

More generally, let the lines \(AD, BE, CF\) intersect at a point \(P\) with trilinear coordinates \((x, y, z)\) with respect to a given reference triangle \(ABC\) with sides \(a, b, c\). If we denote lines through \(D, E, F\) by \(d, e, f\) and assume that these pass through \(Q(u, v, w)\), then it is easy to see that the coordinates of \(d, e, f\) are
\[
[vz - wy, -uz, uy], [vz - uz + wx, -vx], [-wy, wx, uy - vx],
\]
respectively. Since the condition that lines \([l_1, m_1, n_1]\) and \([l_2, m_2, n_2]\) are perpendicular is
\[
l_1l_2 + m_1m_2 + n_1n_2 - (m_1n_2 + m_2n_1) \cos A - (l_1n_2 + n_2l_1) \cos B - (l_1m_2 + m_2l_1) \cos C = 0,
\]
([1], p. 186), we may write
\[
d \perp a \quad \Rightarrow \quad (z \cos C - y \cos B)u + vz - yw = 0,
\]
\[
e \perp b \quad \Rightarrow \quad zu + (z \cos C - x \cos A)v - xw = 0,
\]
\[
f \perp c \quad \Rightarrow \quad yu - xv + (y \cos B - x \cos A)w = 0.
\]
Now, in order for \(Q\) to exist, the determinant of the coefficients of \(u, v, w\) in this system must vanish, i.e.,
\[
(y \cos B - z \cos C)(z \cos C - x \cos A)(x \cos A - y \cos B) + x(z^2 - y^2) \cos A + y(x^2 - z^2) \cos B + z(y^2 - x^2) \cos C = 0. \tag{3}
\]
If \(P = I(1, 1, 1)\), the incentre of \(ABC\), then (3) becomes
\[
(cos B - cos C)(cos C - cos A)(cos A - cos B) = 0,
\]
i.e., \(ABC\) must be isosceles in one of the three ways. In the proposal, \(P = I_A(-1, 1, 1)\), the excentre opposite \(A\), which implies \(B = C\) as required.
One can check that the coordinates of the centroid \((bc, ca, ab)\) and the Gergonne point 
\[
\left( \frac{1}{a(s-a)}, \frac{1}{b(s-b)}, \frac{1}{c(s-c)} \right)
\]
(s the semiperimeter) also satisfy the above system, in which case \(Q\) is the circumcentre and the incentre respectively. Are there any other interesting pairs?

Reference:

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; MARIA ASCENSIÓN LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; and the proposer.

The proposer mentioned that the analogous problem for three interior bisectors was solved by Thébault. This case is contained in solution II above.

* * * * *

**CALL FOR PAPERS – ICME**

There will be two 90-minute sessions at the 7th International Congress on Mathematical Education (ICME) in Quebec City (August 1992) on mathematical competitions. Papers are solicited on this topic which are of general interest to the mathematics education community.

Enquiries and proposals for papers should be sent to

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Please send notice of your desire to present a paper no later than **May 31, 1991**.
Crux Mathematicorum

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CONTENTS

The Olympiad Corner: No. 122 ............................................ R.E. Woodrow 33

Book Review ................................................................. 42

Call For Papers—ICME ..................................................... 42

Problems: 1611–1620 ....................................................... 43

Solutions: 1390, 1430, 1486, 1492–1501, 1503 ................................. 45

Jack Garfunkel .............................................................. 64
THE OLYMPIAD CORNER

No. 122

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow,
Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta,
Canada, T2N 1N4.

It has been some time since we published a problem set from France, and in this
issue we give the problems of the 1988 sitting of the French national competition. Thanks
to Bruce Shawyer, Memorial University of Newfoundland, for sending them in.

COMPOSITION DE MATHÉMATIQUES

Session de 1988

Time: 5 hours

Problem I. Let \( N, p, n \) be non-zero whole numbers. Consider a rectangular
matrix \( T \) having \( n \) lines numbered 1 through \( n \), and \( p \) columns numbered 1 through \( p \). For
\( 1 \leq i \leq n \) and \( 1 \leq k \leq p \), the entry in row \( i \) and column \( k \) is an integer \( a_{ik} \) satisfying
\( 1 \leq a_{ik} \leq N \). Let \( E_i \) be the set of integers appearing in row \( i \).

Answer Question 1 or 2.

Question 1.

In this question two further conditions are imposed on \( T \):

i. for \( 1 \leq i \leq n \), \( E_i \) has exactly \( p \) elements;
ii. for different values of \( i \) and \( j \), the sets \( E_i \) and \( E_j \) are different.

Let \( m \) be the smallest value of \( N \) for which, given values for \( n \) and \( p \), one can form a matrix
\( T \) having the preceding properties.

(a) Calculate \( m \) for \( n = p + 1 \).
(b) Calculate \( m \) for \( n = 10^{30} \) and \( p = 1988 \).
(c) Determine the limit of \( m^p/n \) where \( p \) is fixed and \( n \) tends to infinity.

Question 2.

In this question we replace the two extra conditions of Question 1 by the two
following conditions:

i. \( p = n \);
ii. for every ordered pair of positive integers \((i, k)\) with \( i + k \leq n \), the integer \( a_{ik} \)
does not belong to the set \( E_{i+k} \).

(a) Show that for distinct \( i \) and \( j \) the sets \( E_i \) and \( E_j \) are different.
(b) Show that if \( n \) is at least \( 2^q \), where \( q \) is a positive integer, then \( N \geq q + 1 \).
(c) Suppose that \( n = 2^r - 1 \), where \( r \) is a fixed positive integer. Show that \( N \geq r \).

Exercise II. Determine, for \( n \) a positive integer, the sign of \( n^6 + 5n^5 \sin n + 1 \). For
which positive integers \( n \) is it true that

\[
\frac{n^2 + 5n \cos n + 1}{n^6 + 5n^5 \sin n + 1} \geq 10^{-4}?
\]
**Exercise III.** Consider two spheres $\Sigma_1$ and $\Sigma_2$ and a straight line $\Delta$ which does not meet them. For $i = 1$ and $i = 2$, let $C_i$ be the centre of $\Sigma_i$, $H_i$ the orthogonal projection of $C_i$ on $\Delta$, $r_i$ the radius of $\Sigma_i$, and let $d_i$ be the distance of $C_i$ to $\Delta$. Let $M$ be a point on $\Delta$, and for $i = 1$ and $i = 2$, let $T_i$ be the point of contact with $\Sigma_i$ of a plane tangent to $\Sigma_i$ and passing through $M$; set $\delta_i(M) = MT_i$. Situate $M$ on $\Delta$ so that $\delta_1(M) + \delta_2(M)$ is minimized.

**Exercise IV.** Consider five points $M_1, M_2, M_3, M_4, M$ situated on a circle $C$ in the plane. Show that the product of the distances of $M$ from the lines $M_1M_2$ and $M_3M_4$ equals the product of the distances from $M$ to the lines $M_1M_3$ and $M_2M_4$. What can one deduce about $2n + 1$ distinct points $M_1, \ldots, M_{2n}, M$ situated on $C$?

Now we turn to solutions sent in response to the appeal to help “tidy up” the archives.

**K797.** [1983: 270] *Problems from Kvant.*

It is well known that the last digit of the square of an integer is one of the following: 0, 1, 4, 5, 6, 9. Is it true that any finite sequence of digits may appear before the last one, that is, for any sequence of $n$ digits $\{a_1, a_2, \ldots, a_n\}$ there exists an integer whose square ends with the digits $a_1a_2 \ldots a_n b$, where $b$ is one of the digits listed above?

* Solutions by Andy Liu, University of Alberta and by Richard K. Guy, University of Calgary.*

The answer is no. The sequence of digits 101 cannot precede the units digit in any square. Suppose on the contrary that there exist integers $k$ and $y$, and a digit $x$ in $\{0, 1, \ldots, 9\}$, such that $k^2 = 10^4y + 1010 + x$. We know that $k^2 \equiv 0$ or 1 mod 4. Now $10^4y + 1010 + x \equiv 2 + x$ mod 4. Hence $x = 2, 3, 6$ or 7. However, $k^2$ cannot end in 2, 3, or 7. Hence $k^2 = 10^4y + 1016$. We must have $k = 2h$ for some integer $h$, and then $h^2 = 2500y + 254 \equiv 2$ mod 4, a contradiction.

Alternatively, consider any sequence of the form $40k + 39$. Squares cannot end as $\ldots 90, \ldots 91, \ldots 94, \ldots 95$ or $\ldots 99$, and $400k + 396 = (10x \pm 4)^2$ gives $20k + 19 = 5x^2 \pm 4x$, which implies that $x$ is odd and $19 \equiv 5$ mod 4, a contradiction.

* Turning now to the 1985 problems from the Corner, I would like to thank Murray Klamkin who sent in a large number of solutions as well as references to some solutions which have appeared in his book *International Mathematical Olympiads 1979–1985*, MAA, Washington, D.C., 1986. Here is a listing of those problems from the 1985 numbers for which solutions are discussed in his book.

<table>
<thead>
<tr>
<th>Number</th>
<th>Listing</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>p.38/#16</td>
<td>Listed as G.I./5</td>
<td>on pp. 13, 88-91.</td>
</tr>
<tr>
<td>p.38/#21</td>
<td>Listed as S.G./2</td>
<td>on pp. 12, 82.</td>
</tr>
<tr>
<td>p.72/#40</td>
<td>Listed as G.I./1</td>
<td>on pp. 12, 83-84.</td>
</tr>
<tr>
<td>p.72/#41</td>
<td>Listed as P.G./4</td>
<td>on pp. 12, 80-81.</td>
</tr>
</tbody>
</table>
The next block of solutions are for problems from the unused IMO proposals given in the 1985 numbers of Crux.

17. [1985: 38] Proposed by Poland.
Given nonnegative real numbers $x_1, x_2, \ldots, x_k$ and positive integers $k$, $m$, $n$ such that $km \leq n$, prove that

$$n \left( \prod_{i=1}^{k} x_i^m - 1 \right) \leq m \sum_{i=1}^{k} (x_i^n - 1).$$

Solution by Murray S. Klamkin, University of Alberta.
Let $P = x_1 x_2 \cdots x_k$. Since $\sum_{i=1}^{k} x_i^n \geq k P^{n/k}$, it suffices to show that

$$nP^m - n \leq mk P^{n/k} - mk$$
or that

$$\frac{P^m - 1}{m} \leq \frac{P^r - 1}{r}$$

where $r = n/k \geq m$. (1) follows from the known result that $(P^x - 1)/x$ is increasing in $x$ for $x \geq 0$. A proof follows immediately from the integral representation

$$\frac{P^x - 1}{x \ln P} = \int_{0}^{1} e^{xt} \ln P \, dt.$$

Finally it is to be noted that $m$ and $n$ need not be positive integers, just being positive reals with $n \geq km$ suffices.

23. [1985: 39] Proposed by the USSR.
A tetrahedron is inscribed in a unit sphere. The tetrahedron is such that the center of the sphere lies in its interior. Show that the sum of the edge lengths of the tetrahedron exceeds 6.
Comment by Murray S. Klamkin, University of Alberta.

It is shown more generally in [1] that “The total edge length of a simplex inscribed in a unit sphere in $E^n$ with the center in the interior is greater than $2n$”. Also, still more general results are given.

Reference:

25. [1985: 39] Proposed by the USSR.

A triangle $T_1$ is constructed with the medians of a right triangle $T$. If $R_1$ and $R$ are the circumradii of $T_1$ and $T$, respectively, prove that $R_1 > 5R/6$.

Solution by Murray S. Klamkin, University of Alberta.

The inequality should be $R_1 \geq 5R/6$ since equality occurs for an isosceles right triangle. If $a, b, c$ are the sides of $T$ with $c^2 = a^2 + b^2$, then

$$4m_a^2 = 4b^2 + a^2, \quad 4m_b^2 = 4a^2 + b^2, \quad 4m_c^2 = a^2 + b^2,$$

where $m_a$, $m_b$, $m_c$ are the medians corresponding to $a$, $b$, $c$ respectively. Also,

$$R_1 = \frac{m_am_bm_c}{4F_1} = \frac{m_am_bm_c}{3F}$$

where $F = ab/2$ is the area of $T$, $F_1$ the area of $T_1$. Also $R = c/2$. The given inequality now becomes $8m_am_bm_c \geq 5abc$ or, squaring,

$$(a^2 + b^2)(a^2 + 4b^2)(4a^2 + b^2) = 64(m_am_bm_c)^2 \geq 25a^2b^2c^2 = 25a^2b^2(a^2 + b^2).$$

Expanding out and factoring, we obtain the obvious inequality

$$(a^2 + b^2)(a^2 - b^2)^2 \geq 0.$$

There is equality if and only if $a = b$.

48. [1985: 73] Proposed by the USSR.

Let $O$ be the center of the axis of a right circular cylinder; let $A$ and $B$ be diametrically opposite points in the boundary of its upper base; and let $C$ be a boundary point of its lower base which does not lie in the plane $OAB$. Show that

$$\angle BOC + \angle COA + \angle AOB = 2\pi.$$

Correction and solution by Murray S. Klamkin, University of Alberta.

The inequality should read $\angle BOC + \angle COA + \angle AOB \leq 2\pi$, and is immediate by the known result that the sum of the face angles of a convex polyhedral angle is less than $2\pi$. See, e.g., p. 61 of my book USA Mathematical Olympiads 1972–1986, MAA, 1988.
The last seven solutions are to problems from other Olympiads which appeared in the 1985 issues.


In the given diagram, $B$ is a battery, $L$ is a lamp, and $S_1$, $S_2$, $S_3$, $S_4$, $S_5$ are switches. The probability that switch $S_3$ is on is $2/3$, and it is $1/2$ for the other four switches. These probabilities are independent. Compute the probability that the lamp is on.

Solution by Murray S. Klamkin, University of Alberta.

More generally, let $p_i$ and $q_i$ denote switch $S_i$ being on or off, respectively, and also their probabilities so $0 \leq p_i, q_i \leq 1$, $p_i + q_i = 1$. The probability that the lamp is on is given by

$$p_1p_2p_3p_4p_5 + \sum q_1p_2p_3p_4p_5 + \sum q_1q_2p_3p_4p_5 - q_2q_5p_1p_3p_4 - q_1q_4p_2p_3p_5$$

$$+ p_1p_2q_3q_4q_5 + p_4p_5q_1q_2q_3.$$

The sums are symmetric over the $p$'s and $q$'s. Note that if at most one switch is off, the lamp is on; if any two switches except $q_2q_5$ or $q_1q_4$ are off, the lamp is on, and finally if only two switches are on they must be $p_1p_2$ or $p_4p_5$. [Editor's note: this gives a probability of $25/48$ for the above problem.]

*  


A plane is partitioned into an infinite set of unit squares by parallel lines. A triangle $ABC$ is constructed with vertices at line intersections. Show that if $|AB| > |AC|$, then $|AB| - |AC| > 1/p$, where $p$ is the perimeter of the triangle. (Grades 8, 9, 10)

Solution by Murray S. Klamkin, University of Alberta.

Without loss of generality we can take the rectangular coordinates of $A, B, C$ to be $(0, 0)$, $(x, y)$, $(u, v)$, respectively, where $x, y, u, v$ are integers. Then letting $c^2 = |AB|^2 = x^2 + y^2$, $b^2 = |AC|^2 = u^2 + v^2$, $a^2 = |BC|^2 = (x - u)^2 + (y - v)^2$, we have to show that

$$(c - b)(c + b + a) = c^2 - b^2 + (c - b)a > 1.$$ 

Finally, since $c^2 - b^2$ is a positive integer it is $\geq 1$; also $(c - b)a > 0$.


In a convex quadrilateral the sum of the distances from any point within the quadrilateral to the four straight lines along which the sides lie is constant. Show that the quadrilateral is a parallelogram. (Grade 9)
Solution by Murray S. Klamkin, University of Alberta.

Assume for the quadrilateral \( ABCD \) that lines \( AD \) and \( BC \) are not parallel; so they must meet in a point \( E \) as in the figure. Now draw a chord \( e = GH \) of \( ABCD \) perpendicular to the angle bisector of angle \( E \). It follows by considering the area of the isosceles triangle \( EGH \) that the sum of the distances from any point on \( e \) to lines \( AD \) and \( BC \) is constant. Consequently, by hypothesis, the sum of the distances from any point of \( e \) to lines \( AB \) and \( CD \) is constant. If \( AB \) and \( CD \) are not parallel, let their point of intersection be \( F \) and draw 2 chords \( f \) and \( f' \) of \( ABCD \) perpendicular to the angle bisector of angle \( F \), with \( f \) nearer to \( F \). It follows easily that the sum of the perpendiculars to lines \( AB \) and \( CD \) from a point on \( f \) is less than from a point on \( f' \). Consequently, \( AB \) must be parallel to \( CD \). Proceeding in a similar way using another chord \( c' \) parallel to \( c \), it follows also that \( AD \) must be parallel to \( BC \), whence the figure must be a parallelogram.

It is to be noted that a non-convex quadrilateral cannot have the given property. Also as a rider, show that if the sum of the six distances from any point within a hexahedron is constant, the hexahedron must be a parallelepiped.

\[ \text{5. [1985: 239] 1984 Bulgarian Mathematical Olympiad.} \]

Let \( 0 \leq x_i \leq 1 \) and \( x_i + y_i = 1 \), for \( i = 1, 2, \ldots, n \). Prove that

\[ (1 - x_1 x_2 \cdots x_n)^m + (1 - y_1^m)(1 - y_2^m) \cdots (1 - y_n^m) \geq 1 \]

for all positive integers \( m \) and \( n \).

Comment by Murray S. Klamkin, University of Alberta.

In my editorial note to Problem 68-1 (SIAM Review 11(1969) 402-406), I gave the more general inequality

\[ \prod_{i=1}^{n} \left( 1 - \prod_{j=1}^{m} p_{ij} \right) + \prod_{j=1}^{m} \left( 1 - \prod_{i=1}^{n} q_{ij} \right) > 1 \]

(1)

where \( p_{ij} + q_{ij} = 1 \), \( 0 < p_{ij} < 1 \), and \( m, n \) are positive integers greater than 1. (The cases \( m, n = 1 \) are trivial.) Just let \( p_{ij} = x_j \) for all \( i \) and interchange \( m \) and \( n \) to get the given inequality. Other particularly nice special cases of the above inequality are

\[ \left( 1 - \frac{1}{2^n} \right)^m + \left( 1 - \frac{1}{2^m} \right)^n > 1 \]

and

\[ \frac{1}{2^m} + \frac{1}{2^{1/m}} < 1. \]

Further extensions of (1) have been given by Joel Brenner.
If \( \mathbf{a} \) and \( \mathbf{b} \) are given nonparallel vectors, solve for \( x \) in the equation

\[
\frac{\mathbf{a}^2 + x \mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{a} + x \mathbf{b}|} = \frac{\mathbf{b}^2 + \mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}||\mathbf{a} + \mathbf{b}|}.
\]

Solution by Murray S. Klamkin, University of Alberta.
One can solve by squaring out and solving the rather messy quadratic in \( x \). However, it is easier to proceed geometrically. Referring to the figure, the given equation requires that \( \angle SPQ = \angle PRQ \) (= \( \phi \), say). Thus \( \triangle PQS \sim \triangle RQP \).

![Diagram](image)

Thus, putting \( a = |\mathbf{a}| \) and \( b = |\mathbf{b}| \), \( xb/a = a/b \) or \( x = a^2/b^2 \).

Also, \( x = -x' \) can be negative as in the following figure.

![Diagram](image)

By the law of sines, \( a/\sin(\theta - \phi) = x'b/\sin \phi \) and \( a/\sin \phi = b/\sin(\theta + \phi) \). Hence,

\[
\sin(\theta + \phi) - \sin(\theta - \phi) = \frac{b\sin \phi}{a} - \frac{a\sin \phi}{x'b}
\]

or \( 2\cos \theta = b/a - a/x'b \). Finally,

\[
x = -x' = \frac{-a^2}{b(b - 2a \cos \theta)} = \frac{-a^2}{b^2 + 2a \cdot b}.
\]

*

Determine the maximum value of

\[
\sin^2 \theta_1 + \sin^2 \theta_2 + \cdots + \sin^2 \theta_n,
\]

where \( \theta_1 + \theta_2 + \cdots + \theta_n = \pi \) and all \( \theta_i \geq 0 \).

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let \( f(n) \) denote the maximum value. Then obviously \( f(1) = 0 \) and \( f(2) = 2 \), the latter attained if and only if \( \theta_1 = \theta_2 = \pi/2 \). When \( n = 3 \), it is well known that

\[
\sin^2 \theta_1 + \sin^2 \theta_2 + \sin^2 \theta_3 \leq \frac{9}{4}
\]

with equality just when \( \theta_1 = \theta_2 = \theta_3 = \pi/3 \). (See O. Bottema et al, Geometric Inequalities, item 2.3 on p. 18 or the equivalent inequality in item 2.23 on p. 25.) Thus \( f(3) = 9/4 \).

We show that \( f(n) = 9/4 \) for all \( n \geq 3 \) by proving the following theorem.

**Theorem:** Let \( \theta_i \geq 0 \) for all \( i = 1, 2, \ldots, n \) be such that \( \sum_{i=1}^{n} \theta_i = \pi \) with \( n \geq 3 \). Then \( \sin^2 \theta_1 + \sin^2 \theta_2 + \cdots + \sin^2 \theta_n \leq 9/4 \) with equality holding if and only if \( \theta_1 = \theta_2 = \theta_3 = \pi/3 \) and \( \theta_i = 0 \) for all \( i = 4, 5, \ldots, n \), where we have, without loss of generality, renumbered the indices so that \( \theta_1 \geq \theta_2 \geq \theta_3 \geq \ldots \geq \theta_n \).

We first give a simple lemma.

**Lemma:** If \( A \geq 0 \) and \( B \geq 0 \) are such that \( A + B \leq \pi/2 \), then \( \sin^2 A \sin^2 B \leq \sin^2(A + B) \) with equality if and only if \( A = 0 \) or \( B = 0 \) or \( A + B = \pi/2 \).

**Proof:** \[
\sin^2(A + B) - (\sin^2 A + \sin^2 B)
= \sin^2 A \cos^2 B + 2 \sin A \sin B \cos A \cos B + \cos^2 A \sin^2 B - \sin^2 A - \sin^2 B
= 2 \sin A \sin B \cos A \cos B - \sin^2 A (1 - \cos^2 B) - \sin^2 B (1 - \cos^2 A)
= 2 \sin A \sin B (\cos A \cos B - \sin A \sin B)
= 2 \sin A \sin B \cos(A + B) \geq 0. \quad \Box
\]

To prove the theorem, we use induction on \( n \). The case \( n = 3 \) is the classical result mentioned above. Suppose the theorem holds for some \( n \geq 3 \) and suppose \( \theta_i \geq 0 \) with \( \sum_{i=1}^{n+1} \theta_i = \pi \) and \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n+1} \). Let \( \phi_i = \theta_i \) for \( i = 1, 2, \ldots, n-1 \) and \( \phi_n = \theta_n + \theta_{n+1} \). Then \( \sum_{i=1}^{n} \phi_i = \pi \) and \( \phi_n \leq \pi/2 \), for otherwise \( \theta_n + \theta_{n+1} > \pi/2 \) together with \( n \geq 3 \) would imply that \( \sum_{i=1}^{n+1} \theta_i > \pi \), a contradiction. Hence the lemma and the induction hypothesis we obtain

\[
\sum_{i=1}^{n+1} \sin^2 \theta_i = \sum_{i=1}^{n} \sin^2 \phi_i + \sin^2 \theta_n + \sin^2 \theta_{n+1} \leq \sum_{i=1}^{n} \sin^2 \phi_i \leq \frac{9}{4}.
\]

If equality holds it must hold in the lemma. This implies that either \( \theta_{n+1} = 0 \) or \( \theta_n + \theta_{n+1} = \pi/2 \). However, if \( \theta_n + \theta_{n+1} = \pi/2 \), then \( \theta_1 + \theta_2 \geq \pi/2 \) and since \( \sum_{i=1}^{n+1} \theta_i = \pi \), we obtain \( \theta_1 + \theta_2 = \pi/2 \) and \( n = 4 \). Since \( \theta_1 \geq \theta_2 \geq \theta_3 \geq \theta_4 \), it follows immediately that \( \theta_i = \pi/4 \), \( i = 1, 2, 3, 4 \), from which we obtain \( \sum_{i=1}^{4} \sin^2 \theta_i = 2 \), a contradiction. Thus

Prove that

\[
\frac{x_1^2}{x_1^2 + x_2x_3} + \frac{x_2^2}{x_2^2 + x_3x_4} + \cdots + \frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_1} + \frac{x_n^2}{x_n^2 + x_1x_2} \leq n - 1,
\]

where all \( x_i > 0 \).

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let \( S_n \) denote the cyclic sum. Clearly

\[
S_2 = \frac{x_1^2}{x_1^2 + x_2x_1} + \frac{x_2^2}{x_2^2 + x_1x_2} = 1.
\]

We prove by induction that \( S_n < n - 1 \) for all \( n \geq 3 \). Consider the case \( n = 3 \). Since the sum is cyclic and since the expression is unchanged when \( x_2 \) and \( x_3 \) are interchanged, we may assume that \( x_1 \leq x_2 \leq x_3 \). Then

\[
S_3 = \frac{x_1^2}{x_1^2 + x_2x_3} + \frac{x_2^2}{x_2^2 + x_3x_1} + \frac{x_3^2}{x_3^2 + x_1x_2} < \frac{x_1^2}{x_1^2 + x_2x_1} + \frac{x_2^2}{x_2^2 + x_2x_1} + 1 = 2.
\]

Now suppose \( S_n < n - 1 \) for some \( n \geq 3 \) and consider \( S_{n+1} \) for \( n + 1 \) positive numbers \( x_1, x_2, \ldots, x_{n+1} \). Without loss of generality, we may assume that \( x_{n+1} = \max\{x_i : i = 1, 2, \ldots, n+1\} \). Note that

\[
S_{n+1} = S_n + \frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_{n+1}} + \frac{x_n^2}{x_n^2 + x_{n+1}x_1} + \frac{x_{n+1}^2}{x_{n+1}^2 + x_1x_2} - \frac{x_{n-1}^2}{x_{n-1}^2 + x_{n+1}x_1} - \frac{x_n^2}{x_n^2 + x_1x_2}.
\]

Since

\[
\frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_{n+1}} \leq \frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_1}, \quad \frac{x_n^2}{x_n^2 + x_{n+1}x_1} \leq \frac{x_n^2}{x_n^2 + x_1x_2}, \quad \text{and} \quad \frac{x_{n+1}^2}{x_{n+1}^2 + x_1x_2} < 1,
\]

we conclude that \( S_{n+1} < S_n + 1 < n \), completing the induction.

Remark. If we set \( x_i = t^i \) for \( i = 1, 2, \ldots, n - 1 \), and \( x_n = t_{2n} \), then

\[
S_n = \frac{t^2}{t^2 + t^5} + \frac{t^4}{t^4 + t^7} + \cdots + \frac{t^{2n-4}}{t^{2n-4} + t^{3(n-1)}} + \frac{t^{2n-2}}{t^{2n-2} + t^{2n+1}} + \frac{t^{4n}}{t^{4n} + t^3}
\]

\[\to n - 1 \text{ as } t \to 0^+.
\]

That is, though the bound \( n - 1 \) cannot be attained for \( n > 2 \), it is nonetheless sharp.

\[
* \quad * \quad * \quad *
\]

This completes the “archive” material for 1985 and the space available this number.

Contest season is upon us. Send me your contests and solutions!
BOOK REVIEW


This report on the 1989 IMO first lists the rules and regulations governing the competition. The program of events comes next followed by the competition problems and solutions. All 109 proposed problems are listed next with solutions given to the 32 that were shortlisted by the jury. (These 32 problems have appeared in Crux in [1989: 196–197, 225–226, 260–262].) The contestants' scores and awards are given last. The book is an excellent source of problems, but it would have been nice to see the solutions given to more of the difficult problems which were not shortlisted. For example, I have yet to see a good proof (one reasonably explained and under four pages long) of the following proposed question:

HEL 2. In a triangle $ABC$ for which $6(a+b+c)r^2 = abc$ holds and where $r$ denotes the inradius of $ABC$, we consider a point $M$ of its inscribed circle and projections $D, E, F$ of $M$ on the sides $BC, AC, AB$, respectively. Let $S$ and $S_1$ denote the areas of the triangles $ABC$ and $DEF$, respectively. Find the maximum and minimum values of the quotient $S/S_1$.

* * * * *

CALL FOR PAPERS – ICME

There will be two 90-minute sessions at the ICME in Quebec City (August 1992) on mathematical competitions. Papers are solicited on this topic which are of general interest to the mathematics education community.

Enquiries and proposals for papers should be sent to

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Please send notice of your desire to present a paper no later than May 31, 1991.
PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1991, although solutions received after that date will also be considered until the time when a solution is published.

1611. Proposed by George Tsintsifas, Thessaloniki, Greece.
Let $ABC$ be a triangle with angles $A, B, C$ (measured in radians), sides $a, b, c$, and semiperimeter $s$. Prove that

\[ (i) \sum \frac{b + c - a}{A} \geq \frac{6s}{\pi}; \quad (ii) \sum \frac{b + c - a}{aA} \geq \frac{9}{\pi}. \]

1612*. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.
Let $x, y, z$ be positive real numbers. Show that

\[ \sum \frac{y^2 - x^2}{z + x} \geq 0, \]
where the sum is cyclic over $x, y, z$, and determine when equality holds.

1613. Proposed by Murray S. Klamkin, University of Alberta.
Prove that

\[ \left( \frac{\sin x}{x} \right)^2 + \left( \frac{\tan x}{x} \right)^p \geq 2 \]

for $p \geq 0$ and $0 < x < \pi/2$. (The case $p = 1$ is problem E3306, American Math. Monthly, solution in March 1991, pp. 264–267.)

1614. Proposed by Toshio Seimiya, Kawasaki, Japan.
Let $D$ and $E$ be points on side $BC$ of a triangle $ABC$. Draw lines through $D, E$ parallel to $AC, AB$ respectively, meeting $AB$ and $AC$ at $F$ and $G$. Let $P$ and $Q$ be the intersections of line $FG$ with the circumcircle of $\triangle ABC$. Prove that $D, E, P$ and $Q$ are concyclic.
1615. Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.

Consider the following array:

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \ldots \\
4 & 2 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots \\
6 & 2 & 7 & 4 & 8 & 9 & 10 & 11 & 12 & 13 & \ldots \\
8 & 7 & 9 & 2 & 6 & 11 & 12 & 13 & 14 & \ldots \\
6 & 2 & 11 & 9 & 12 & 7 & 13 & 8 & 14 & 15 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

For example, to produce row 5 from row 4, write down, in order: the 1st number to the right of 4, the 1st number to the left of 4, the 2nd to the right of 4, the 2nd to the left of 4, the 3rd to the right of 4, the 3rd to the left of 4, and then the 4th, 5th, \ldots numbers to the right of 4.

Notice that a number will be expelled from a row if and only if it is the diagonal element in the previous row (the boxed numbers in the array), and once missing it of course never reappears.

(a) Is 2 eventually expelled?

(b) Is every positive integer eventually expelled?


In triangle $ABC$, angles $\alpha$ and $\gamma$ are acute, $D$ lies on $AC$ so that $BD \perp AC$, and $E$ and $F$ lie on $BC$ so that $AE$ and $AF$ are the interior and exterior bisectors of $\angle BAC$. Suppose that $BC$ is the exterior bisector of $\angle ABD$. Show that $AE = AF$.


If $p$ is a prime and $a$ and $k$ are positive integers such that $p^k | (a - 1)$, then prove that

\[
p^{n+k} | (a^n - 1)
\]

for all positive integers $n$.

1618. Proposed by Jordan Tabov, Sofia, Bulgaria. (Dedicated to Dan Pedoe on the occasion of his 80th birthday.)

The sides of a given angle $\alpha$ intersect a given circle $O(r)$ in four points. Four circles $O_k(r_k)$, $k = 1, 2, 3, 4$, are inscribed in $\alpha$ so that $O_1(r_1)$ and $O_4(r_4)$ touch $O(r)$ externally, and $O_2(r_2)$ and $O_3(r_3)$ touch $O(r)$ internally. Prove that $r_1r_4 = r_2r_3$. (This is more general than problem 1.8.9, page 20, of Japanese Temple Geometry Problems by H. Fukagawa and D. Pedoe.)

1619. Proposed by Hui-Hua Wan and Ji Chen, Ningbo University, Zhejiang, China.

Let $P$ be an interior point of a triangle $ABC$ and let $R_1, R_2, R_3$ be the distances from $P$ to the vertices $A, B, C$, respectively. Prove that, for $0 < k < 1$,

\[
R_1^k + R_2^k + R_3^k < (1 + 2^{1/k})^{1-k}(a^k + b^k + c^k).
\]
1620. Proposed by Ilia Blaskov, Technical University, Gabrovo, Bulgaria.
Find a finite set $S$ of (at least two) points in the plane such that the perpendicular bisector of the segment joining any two points in $S$ passes through exactly two points of $S$.

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


$A, B, C$ are points on a circle, such that $CM$ is the perpendicular bisector of $AB$ [where $M$ lies on $AB$]. $P$ is a point on $CM$ and $AP$ meets, again at $D$. As $P$ varies over segment $CM$, find the largest radius of the inscribed circle tangent to segments $PD$, $PB$, and arc $DB$ of, in terms of the length of $CM$.

II. Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

Let $(O, R) =$, be the circumcircle of $\Delta ABC$, and $(Q, \rho)$ the given inscribed circle.
We denote

$$\varphi = \angle PAB = \angle PBA = \angle BPQ = \angle SPQ,$$

$S$ being the projection of $Q$ to $PD$. Let $MA = MB = x$, $OM = d$, $PM = m$. $C'$ is the second intersection point of $CP$ and $PD$.

In right triangle $OPQ$ we have

$$PQ = \frac{\rho}{\sin \varphi}, \quad OP = m - d, \quad OQ = R - \rho$$

and so

$$\frac{\rho^2}{\sin^2 \varphi} + (m - d)^2 = (R - \rho)^2,$$

or

$$\rho^2 \cot^2 \varphi + 2R\rho - R^2 + (m - d)^2 = 0. \quad (1)$$

Now

$$\cot^2 \varphi = \frac{\rho^2}{m^2} = \frac{R^2 - d^2}{m^2},$$

and (1) becomes

$$\rho^2(R^2 - d^2) + 2Rm^2\rho - m^2[R^2 - (m - d)^2] = 0. \quad (2)$$
It is clear that $R > m - d$, so $R^2 - (m - d)^2 > 0$, and we conclude that (2) has two different roots $\rho_1 > 0$ and $\rho_2 < 0$. $D$ being the discriminant of (2) we have

$$
\frac{D}{4} = R^2 m^4 + m^2 (R^2 - d^2) [R^2 - (m - d)^2] = m^2 (R^4 - 2R^2 d^2 + d^4 + 2R^2 dm - 2d^3 m + d^2 m) = m^2 (R^2 - d^2 + dm)^2.
$$

So

$$
\rho_1 = \frac{-R m^2 + m (R^2 - d^2 + dm)}{R^2 - d^2} = \frac{m (R^2 - d^2) - m^2 (R - d)}{R^2 - d^2} = \frac{m (R + d) - m^2}{R + d} = \frac{m (R + d - m)}{R + d},
$$

and thus

$$
\frac{1}{\rho_1} = \frac{R + d}{m (R + d - m)} = \frac{1}{m} + \frac{1}{R + d - m} = \frac{1}{PM} + \frac{1}{PC},
$$

(3) the “harmonic mean” property given on [1990: 27], with a maximum for $\rho_1$ when $PM = PC$. Also

$$
\rho_2 = \frac{-R m^2 - m (R^2 - d^2 + dm)}{R^2 - d^2} = \frac{-m (R^2 - d^2) - m^2 (R + d)}{R^2 - d^2} = \frac{-m (R - d) - m^2}{R - d} = \frac{-m (R - d + m)}{R - d},
$$

so

$$
\frac{1}{\rho_2} = \frac{1}{R - d + m} - \frac{1}{m} = \frac{1}{PC'} - \frac{1}{PM'},
$$

(4) From (3) and (4) follows

$$
\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{PC'} + \frac{1}{PC}.
$$

$\rho_2$ has a geometrical meaning: $-\rho_2 > 0$ is the radius of the circle touching the products of $AP$ and $PB$, and touching, externally. In the figure, $(Q', \rho')$ is this circle. We have

$$
\frac{\rho'^2}{\sin^2 \phi} + (m - d)^2 = (R + \rho')^2
$$

or (as above)
\[ \rho^2(R^2 - d^2) - 2Rm^2 \rho' - m^2[R^2 - (m - d)^2] = 0. \]  

Comparing (2) and (5) we conclude: if (2) has the roots \( \rho_1 \) and \( \rho_2 \), then (5) has the roots \(-\rho_1\) and \(-\rho_2\).


We use the following result, due to Poncelet (e.g., [2]):

**THEOREM.** Let points \( A, B, C, D \) lie on one circle \( O \) of a coaxial family of circles, and suppose that a second circle \( I \) of the family touches lines \( AB \) and \( CD \). Then there are circles \( J \) and \( K \) of the family, \( J \) touching \( AC \) and \( BD \), and \( K \) touching \( AD \) and \( BC \). Moreover the six contact points of \( I, J, K \) with these lines are collinear. \( \square \)

[Editor’s note: expert colleague Chris Fisher, University of Regina, also suggests the reference [1].]

In the present problem we draw the parallel to \( AB \) through \( D \), letting it meet \( CM \) at \( M' \) and again at \( D' \), and let \( I \) (with centre \( I \)) be the circle tangent to these parallels and internally tangent to \( C \), at a point \( L \) in the minor arc \( BD \). The coaxial family containing \( C \), and \( I \) is thus the family of all circles mutually tangent at \( L \). By the above theorem of Poncelet, the incircle \((Q, \rho)\) tangent to \( AD, BD', \) and minor arc \( BD \) of \( C \), will be tangent to \( C \), at \( L \). A fourth circle, exterior to \( C \), will be tangent to \( C \), at \( L \) and also to \( AD' \) and \( BD \).

Let \( E \) be the midpoint of \( AD \), so that \( OE \perp AD \) and \( EI \perp MM' \), and \( EI \) meets \( MM' \) at its midpoint \( N \). Let \( CC' \) be a diameter of \( C \). Put \( AP = a, CP = c, MP = m \). Then \( C'M + m = C'P = R + OP \), or

\[ C'M - OP = R - m. \]  

We have

\[ (AM)^2 = CM \cdot C'M \]  

so that

\[ \frac{ON}{OP} = \frac{ON}{OE} \cdot \frac{OE}{OP} = \left( \frac{AM}{AP} \right)^2 = \frac{CM \cdot C'M}{a^2}, \]  

or

\[ a^2 ON = OP \cdot CM \cdot C'M. \]  

Also

\[ \frac{ON}{OP} = \frac{OI}{OQ} = \frac{R - IL}{R - QL} = \frac{R - MN}{R - \rho}, \]  

\[ \frac{ON}{OP} = \frac{OI}{OQ} = \frac{R - IL}{R - QL} = \frac{R - MN}{R - \rho}, \]
and thus from (3),
\[
\frac{R - MN}{R - \rho} = \frac{CM \cdot C'M}{a^2}.
\] (5)
Next \( R + ON = C'N = C'M + MN \), and so
\[
R - MN = C'M - ON.
\] (6)
Now we have from (5),(6),(4),(2) and (1),
\[
CM \cdot C'M \cdot (R - \rho) = a^2(R - MN) = a^2(C'M - ON)
= C'M(a^2 - OP \cdot CM) = C'M(m^2 + (AM)^2 - OP \cdot CM)
= C'M(m^2 + CM(C'M - OP)) = C'M(R \cdot CM - (CM - m)m)
= C'M(R \cdot CM - cm).
\]
Thus \( CM(R - \rho) = R \cdot CM - cm \), or
\[
\rho = \frac{cm}{CM} = \frac{cm}{c + m}.
\]
Therefore the diameter \( 2\rho \) of the incircle is the harmonic mean of \( CP \) and \( MP \) (and, incidentally, also of \( C'P \) and \( M'P \)). The result then follows as before.

References:
[1] Marcel Berger, Geometry II (English translation), Springer, Berlin, 1987, Prop. 16.6.4 and Fig. 16.6.4(2), pp. 203 and 204.

Editor’s note. Walker also observes that, letting \( CQ \) and \( AMB \) meet at \( S \), and \( CL \) and \( PQ \) meet at \( T \), one gets \( MS = PT = PA \), and therefore \( OL, CS, PT \) concur (at \( Q \)). He uses this approach to give a second solution, not using the Poncelet theorem.

A further solution has been received from TOSHIIO SEIMIYA, Kawasaki, Japan. Seimiyia first establishes the following lemma. \( ABC \) is a triangle with incenter \( I \) and circumcircle \( C \). Let \( D \) be a point on side \( BC \). Let \( \omega \) be an inscribed circle tangent to the segments \( AD \) and \( DC \), at \( E \) and \( F \) respectively, and to the arc \( AC \) of \( C \). Then \( E, F \) and \( I \) are collinear. Seimiyia uses this lemma to prove the present problem, as well as to give another proof of Crux 1260 [1988: 237; 1989: 51]. He also points out that the harmonic mean relationship given above occurs as problem 1.2.7, p. 5 of H. Fukagawa and D. Pedoe, Japanese Temple Geometry Problems (reviewed on [1990: 203]).

* * * * *


\( AD, BE, CF \) are (not necessarily concurrent) cevians in triangle \( ABC \), intersecting the circumcircle of \( \triangle ABC \) in the points \( P, Q, R \). Prove that
\[
\frac{AD}{DP} + \frac{BE}{EQ} + \frac{CF}{FR} \geq 9.
\]
When does equality hold?
II. Solution by Ji Chen, Ningbo University, China.

The result may be sharpened to

\[ \frac{AD}{DP} + \frac{BE}{EQ} + \frac{CF}{FR} \geq \left(4 - \frac{2r}{R}\right)^2. \]

From the inequality (given in the published solution of Seimiya [1990: 159])

\[ \frac{AD}{DP} + \frac{BE}{EQ} + \frac{CF}{FR} \geq \sum \left(\frac{(b + c)^2}{a^2} - 1\right) = \sum a \cdot \sum \frac{b + c - a}{a^2}, \]

where the sums are cyclic over \(a, b, c\), and the fact that

\[ 4 - \frac{2r}{R} = 4 - \prod \frac{b + c - a}{a} = \frac{\sum a(b + c - a)^2}{\prod a}, \]

it is enough to show that

\[ \sum a \cdot \sum b^2 c^2 (b + c - a) \geq \left(\sum a(b + c - a)^2\right)^2. \]

Putting \(x = b + c - a, \ y = c + a - b, \ z = a + b - c,\)

this becomes

\[ \sum x \cdot \sum x \left(\frac{x + y}{2}\right)^2 \left(\frac{x + z}{2}\right)^2 \geq \left(\sum x^2 \left(\frac{y + z}{2}\right)\right)^2, \]

or

\[ \sum x \cdot \sum x(x + y)^2 (x + z)^2 - 4\sum x^2(y + z)^2 \geq 0, \]

where the sums are cyclic over \(x, y, z\). The left side of (1) becomes

\[
\begin{align*}
\sum x^6 & + 3 \sum (x^5 y + x^5 z) - \sum (x^4 y^2 + x^4 z^2) - 6 \sum x^3 y^3 \\
& + 2 \sum (x^3 y^2 z + x^3 z^2 y) - 9 x^2 y^2 z^2 \\
& = \left[\sum x^6 - \sum (x^4 y^2 + x^4 z^2) + 3 x^2 y^2 z^2\right] + 3 \left[\sum (x^5 y + x^5 z) - 2 \sum y^3 z^3\right] \\
& + 2 \left[\sum (x^3 y^2 z + x^3 z^2 y) - 6 x^2 y^2 z^2\right] \\
& = \frac{2 \sum x^2}{2} \\
& + 3 \sum y z(y^2 - z^2)^2 + 2 \sum x^2 y z(y - z)^2 \\
& \geq 0,
\end{align*}
\]

so (1) follows.

Editor’s note. Murray Klamkin has obligingly supplied the editor with the following simpler proof of inequality (1). By Cauchy’s inequality,

\[ \sum x \cdot \sum x(x + y)^2 (x + z)^2 \geq \left(\sum x(x + y)(x + z)\right)^2, \]
so it suffices to prove the stronger inequality
\[ \sum x(x + y)(x + z) \geq 2 \sum x^2(y + z), \]
which simplifies to
\[ \sum x(x - y)(x - z) \geq 0. \] (2)

But assuming without loss of generality that \( x \geq y \geq z \), (2) follows from
\[ x(x - y)(x - z) \geq y(x - y)(y - z) \quad \text{and} \quad z(z - x)(z - y) \geq 0. \]
(2) is the special case \( n = 1 \) of Schur’s inequality
\[ \sum x^n(x - y)(x - z) \geq 0 \]
which can be established the same way (see pp. 49–50 of M.S. Klamkin, *International Mathematical Olympiads 1979–1985*, M.A.A., 1986).

* * * * *


Given three triangles \( T_1, T_2, T_3 \) and three points \( P_1, P_2, P_3 \), construct points \( X_1, X_2, X_3 \) such that the triangles \( X_2X_3P_1, X_3X_1P_2, X_1X_2P_3 \) are directly similar to \( T_1, T_2, T_3 \), respectively.

Solution by the proposer.

Denote by \( S_i \) the dilative rotation with centre \( P_i \), angle \( \gamma_i = \angle A_iC_iB_i \) (in the sense \( C_iA_i \) to \( C_iB_i \)), and ratio \( r_i = C_iB_i/C_iA_i \), for \( i = 1, 2, 3 \). For each point \( X \), the triangle \( XS_i(X)P_i \) will be similar to \( A_iB_iC_i \). Let \( S = S_1S_2S_3 \), i.e. \( S(X) = S_3(S_2(S_1(X))). \) \( S \) is a dilative rotation of angle \( \gamma = \gamma_1 + \gamma_2 + \gamma_3 \), ratio \( r = r_1r_2r_3 \) and with fixed point \( O \), say. If we put \( X_2 = O \), \( S_1(X_2) = X_3 \) and \( S_2(X_3) = X_1 \), then \( S_3(X_1) = S(X_2) = X_2 \). Thus the problem reduces to constructing the fixed point of \( S \) and this point will be \( X_2 \). \( X_3 \) and \( X_1 \) will then be \( S_1(X_2) \) and \( S_3^{-1}(X_2) \) respectively. If \( \gamma \neq 0^\circ \) or \( 360^\circ \), or \( r \neq 1 \), the problem has a unique solution. [Editor’s note. Dou ended with a construction of the required points. A construction of the unique fixed point given three noncollinear points and their images under a similarity can be found in references such as Coxeter’s *Introduction to Geometry*, page 73.]

* * * * *


Let \( A'B'C' \) be a triangle inscribed in a triangle \( ABC \), so that \( A' \in BC, B' \in CA, C' \in AB \). Suppose also that \( BA' = CB' = AC' \).

(a) If either the centroids \( G, G' \) or the circumcenters \( O, O' \) of the triangles coincide, prove that \( \Delta ABC \) is equilateral.
(b) If either the incenters \( I, I' \) or the orthocenters \( H, H' \) of the triangles coincide, characterize \( \Delta ABC \).

I. Solution to part (a) by Murray S. Klamkin, University of Alberta.

Let \( A, B, C \) denote vectors from a common origin to the respective vertices \( A, B, C \). It then follows that

\[ A' = B + \frac{\alpha}{a}(C - B) = \frac{\alpha}{a}C + \left(1 - \frac{\alpha}{a}\right)B \]

and similarly

\[ B' = \frac{\alpha}{b}A + \left(1 - \frac{\alpha}{b}\right)C, \quad C' = \frac{\alpha}{c}B + \left(1 - \frac{\alpha}{c}\right)A, \]

where \( a,b,c \) are the respective sides of \( ABC \) and \( \alpha \) is a nonzero constant.

For the case \( G = G' \), we choose the origin to be the centroid of \( \Delta ABC \) so that \( A + B + C = 0 \) and also

\[ 0 = A' + B' + C' = \left(\frac{\alpha}{b} - \frac{\alpha}{c} + 1\right)A + \left(\frac{\alpha}{c} - \frac{\alpha}{a} + 1\right)B + \left(\frac{\alpha}{a} - \frac{\alpha}{b} + 1\right)C. \]

Thus

\[ 0 = \left(\frac{1}{b} - \frac{1}{c}\right)A + \left(\frac{1}{c} - \frac{1}{a}\right)B + \left(\frac{1}{a} - \frac{1}{b}\right)C = \left(\frac{2}{b} - \frac{1}{c} - \frac{1}{a}\right)A + \left(\frac{1}{b} + \frac{1}{c} - \frac{2}{a}\right)B, \]

and so

\[ \frac{2}{b} - \frac{1}{c} - \frac{1}{a} = 1 + \frac{1}{c} - \frac{2}{a} = 0, \]

which implies \( a = b = c \).

For the case \( O = O' \), we choose the origin to be the circumcenter of \( \Delta ABC \) so that \( |A| = |B| = |C| = R, \) the circumradius. Then also \( |A'|^2 = |B'|^2 = |C'|^2 = R^2 \), or

\[ R^2 = \frac{\alpha^2}{a^2}|C|^2 + \left(1 - \frac{\alpha}{a}\right)^2|B|^2 + 2 \cdot \frac{\alpha}{a} \left(1 - \frac{\alpha}{a}\right)B \cdot C \]

\[ = \frac{\alpha^2}{a^2}R^2 + \left(1 - \frac{\alpha}{a}\right)^2 R^2 + \frac{2\alpha(a - \alpha)}{a^2}R^2 \cos 2A \]

\[ = \left(\frac{2\alpha^2}{a^2} - \frac{2\alpha}{a} + 1\right)R^2 + \frac{2\alpha(a - \alpha)}{a^2}R^2 - \frac{4\alpha(a - \alpha)R^2 \sin^2 A}{a^2} \]

\[ = R^2 - \alpha(a - \alpha), \quad \text{etc.}, \]

or

\[ \alpha(a - \alpha) = R^2 - R^2 = \alpha(b - \alpha) = \alpha(c - \alpha). \]

Hence \( a = b = c \).

II. Solution to part (a) by the proposer.

It is known (Crux 1464 [1990: 282]) that \( G = G' \) if and only if
\[
\frac{BA'}{AC} = \frac{CB'}{B'A} = \frac{AC'}{C'B'}.
\]

Therefore according to our assumption we have \( A'C = B'A = C'B \), and hence that \( \Delta ABC \) is equilateral.

If \( O = O' \), we easily find that the triangles \( OA'B, OB'C, OC'A \) are congruent, so \( \angle OBA' = \angle OCB' \). But also \( \angle OBA' = \angle OCA' \), so \( CO \) bisects \( \angle C \). Similarly we find that \( O = I \), hence \( \Delta ABC \) is equilateral.

Part (a) also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and by D.J. SMEENK, Zaltbommel, The Netherlands.

Part (b) remains unsolved, as did part (c) of the proposer’s earlier problem Crux 1464 [1990: 282]; in fact Janous comments that in trying (b) he “got stuck in quite awkward and disgusting expressions”, the same predicament that befell him in Crux 1464! Looks like these problems need a new idea from someone.

\[ * * * * * \]


Two squares \( ABDE \) and \( ACFG \) are described on \( AB \) and \( AC \) outside the triangle \( ABC \). \( P \) and \( Q \) are on line \( EG \) such that \( BP \) and \( CQ \) are perpendicular to \( BC \). Prove that \( BP + CQ \geq BC + EG \).

When does equality hold?

I. Solution by Jordi Dou, Barcelona, Spain.

Let \( M, N, O \) be the midpoints of \( BC \), \( PQ \), \( EG \), respectively. Let \( A' \) be symmetric to \( A \) with respect to \( O \). We have that \( AE \) is equal and perpendicular to \( BA \), and analogously \( EA' (= AG) \) to \( AC \). Therefore \( AA' = BC \), \( AA' \perp BC \) and \( OE = MA \), \( OE \perp MA \). Let \( L \) on \( MN \) be such that \( AL\parallel OE \). Then \( AO = LN \), and \( ML > MA \) because \( MA \) is perpendicular to \( AL \). Hence \( MN = LN + ML \geq AO + MA \), and

\[ BP + CQ = 2MN \geq 2(AO + MA) = 2(BM + OE) = BC + EG. \]

Equality occurs if and only if \( MN = MA + AO \), i.e., \( MN \) coincides with \( MO \), i.e., \( AB = AC \).
**Note.** Here is a nice property of triangles between two squares with a common vertex:

| The median of one triangle is perpendicular to the base of the other. |

The proof is: a rotation of $90^\circ$ with centre $O$ moves $ABFC$ onto $DGEA$, consequently $AM \perp DE$ and $AN \perp BC$. This property is a "leitmotif" in the above solution and also in my solution of *Crux* 1496.

II. *Solution par C. Festaerts-Hamoir, Brussels, Belgium.*

Construisons, sur $BC$ et du même côté que $A$, le triangle rectangle isocèle $BOC$. La similitude $\sigma_1$ de centre $B$, d’angle $45^\circ$ et de rapport $\sqrt{2}/2$ applique $E$ sur $A$ et $O$ sur $M$. La similitude $\sigma_2$ de centre $C$, d’angle $45^\circ$ et de rapport $\sqrt{2}$ applique $A$ sur $G$ et $M$ sur $O$. $\sigma_2 \circ \sigma_1$ est donc une rotation d’angle $90^\circ$ (le rapport est $\sqrt{2} \cdot \sqrt{2}/2 = 1$) qui applique $E$ sur $G$ et dont le point fixe est $O$. $EOG$ est ainsi un triangle rectangle isocèle.

$M$ étant le milieu de $BC$,

$$BP + CQ = 2MR = 2MO + 2OR = BC + 2OR \geq BC + 2OH = BC + EG.$$ 

L’égalité a lieu si et seulement si $OR = OH$, c’est-à-dire $EG \parallel BC$, le triangle $ABC$ est alors isocèle ($AB = AC$).

*Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSION LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; L. J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinenymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; and the proposer.*

The solutions of Kuczma and the proposer were very similar to solution I.

* * * * *


Three numbers $x, y, z$ are chosen independently at random and uniformly in $[0,1]$. What is the probability that $x, y, z$ can be the lengths of the sides of a triangle whose altitudes are also the sides of some triangle?

*Solution by P. Penning, Delft, The Netherlands.*

The probability is equal to the volume in $x, y, z$-space where triangles meeting the constraint can be found. Since the size of the triangle cannot play a part, the boundaries of the volume must be straight lines passing through $(0,0,0)$. 
First we impose another constraint,

\[ x \leq y \leq z, \]  \hspace{1cm} (1)

thereby reducing the relevant volume by a factor of \(3! = 6\). These numbers form a triangle if

\[ x \geq z - y. \]  \hspace{1cm} (2)

The altitudes are proportional to \(1/x \geq 1/y \geq 1/z\) (the area of the triangle is the proportionality factor), and they form a triangle if \(1/x \leq 1/y + 1/z\), i.e.

\[ x \geq \frac{yz}{y + z}. \]  \hspace{1cm} (3)

The region where conditions (1)-(3) are met is a pyramid, with top at \((0, 0, 0)\) and base in the plane \(z = 1\). (The base is shown in the figure.) With integral calculus the area is easily determined to be

\[ \ln(\sqrt{5} - 1) + \frac{2 - \sqrt{5}}{4}. \]

The probability is then equal to

\[ 6(\text{volume of pyramid}) = 2(\text{area of base}) \]
\[ = 2 \ln(\sqrt{5} - 1) + 1 - \sqrt{5}/2 \]
\[ \approx 0.3058. \]

If only the sides have to form a triangle, then the probability is 0.5.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer. A solution similar to Penning’s, but with a trivial error at the end, was also sent in by C. FESTRAETS-HAMOIR, Brussels, Belgium.

* * * * * * * * *


(a) Show that there exist infinitely many positive integer solutions to

\[ c^2 = \left(\frac{a}{2}\right) - \left(\frac{b}{2}\right). \]

(b)* If \(k > 2\) is an integer, are there infinitely many solutions in positive integers to

\[ c^k = \left(\frac{a}{k}\right) - \left(\frac{b}{k}\right). \]
I. Solution to part (a) by Hayo Ahlborg, Benidorm, Spain.
The equation becomes \( a^2 - a - b^2 + b = 2c^2 \) or
\[
(a + b - 1)(a - b) = 2c^2.
\]
Introducing \( D = a - b \), we find \( D^2 + (2b - 1)D = 2c^2 \), so
\[
b = \frac{c^2 - D - 1}{2}, \quad a = b + D = \frac{c^2 + D + 1}{2}.
\]
To make \( a \) and \( b \) integers, we have two possibilities.

Case (i): \( c^2 = kD \). Then
\[
a = k + \frac{D + 1}{2}, \quad b = k - \frac{D - 1}{2},
\]
where \( D \) is odd and \( k \) is any integer which makes \( kD \) a square. For example \( (D = 1) \):
\[
a = c^2 + 1, \quad b = c^2, \quad c \text{ arbitrary}.
\]  
(1)

Case (ii): \( c^2 = mD + D/2 \). Here \( D = 2^{2n+1}d \) with \( d \) odd, and \( m \) is any number which makes \((2m + 1)d \) a square. In this case we have
\[
c^2 = (2m + 1)2^{2n}d, \quad a = m + 1 + 2^{2n}d, \quad b = m + 1 - 2^{2n}d.
\]
For example \( (D = 2) \):
\[
a = \frac{c^2 + 3}{2}, \quad b = \frac{c^2 - 1}{2}, \quad c \text{ arbitrary odd}.
\]  
(2)

II. Partial solution to part (b) by Kenneth M. Wilke, Topeka, Kansas.
[Wilke first solved part (a)—Ed.]
Consider \( k = 3 \). Then \( c^3 = \left(\begin{array}{c} a \\ 3 \end{array}\right) - \left(\begin{array}{c} b \\ 3 \end{array}\right) \) becomes
\[
6c^3 = (a - 1)^3 - (a - 1) - [(b - 1)^3 - (b - 1)]
= (a - b)(a^2 + ab + b^2 - 3(a + b) + 2).
\]
Taking \( a = b + 2 \), this becomes \( 6c^3 = 2 \cdot 3b^2 \), whence we can take
\[
c = t^2, \quad b = t^3, \quad a = t^3 + 2 \quad (t \text{ arbitrary})
\]
to produce an infinite number of solutions. These are not all the solutions when \( k = 3 \).
Others include \((a, b, c) = (9, 6, 4), (12, 4, 6) \) and \((25, 19, 11) \).

Also solved (both parts) by H. L. ABBOTT, University of Alberta; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; and RICHARD K. GUY.
The remarkable solution II was also found by Engelhaupt and by Guy. Abbott’s solution to part (b) was a recursively defined infinite sequence of solutions, the first being the trivial

\[ 1^3 = \binom{3}{3} - \binom{2}{3}, \]

and the second being the already formidable

\[ 16199^3 = \binom{30527}{3} - \binom{14328}{3}. \]

These all have the additional property that \( a = b + c \). Guy has since found other solutions with this property, for instance

\[ 26^3 = \binom{50}{3} - \binom{24}{3}. \]

Example (1) of case (i) in Ahlburg’s solution was also found by Engelhaupt, Englander and Gibbs, while example (2) of case (ii) was again given by Engelhaupt. Another special case of Ahlburg’s case (i), namely \( k = D(\text{odd}) \), was found by Hess, Kierstead and Klamkin. This gives the nice solution

\[ a = 3b - 1, \quad c = 2b - 1, \quad b \text{ arbitrary.} \]

No solutions for \( k > 3 \) were sent in. However, Guy has come up with the single (and singular!) example

\[ 6^5 = \binom{18}{5} - \binom{12}{5} \]

for \( k = 5 \). Is it part of some infinite family?  

*     *     *     *     *     *     * 


There are four squares \( ABCD, DEFG, AEIJ, BJKL \) as shown in the figure. Show that \( L, C, G \) are collinear if and only if \( 2AD = AE \).
I. Solution by Jordi Dou, Barcelona, Spain.

Let $M$, $N$, $O_1$ be the midpoints of $AE$, $AJ$, $CL$, respectively. $B'$ is symmetric to $B$ with respect to $O_1$, and $O$ is the centre of $ABCD$.

Triangle $CBB'$ is obtained from triangle $BAJ$ by a rotation of $90^\circ$ about centre $O$. Therefore $BN \perp CL$. [Editor’s note: see also Dou’s comment at the end of his solution to Crux 1493, this issue!] Analogously, $DM \perp CG$. If $AE = 2AD$ then $AB = AN = AM = AD$ and $\angle NAM = \angle BAD (= 90^\circ)$, so $BN$ is parallel to $DM$, and therefore $L$, $C$, $G$ are collinear. It is clear that if $AE \neq 2AD$, then $DM$ is not parallel to $BN$ and $L$, $C$, $G$ are not aligned.

II. Solution by Marcin E. Kuczma, Warszawa, Poland.

Consider the vectors $\overrightarrow{AD} = u$, $\overrightarrow{AE} = v$. Let $h$ denote rotation by $90^\circ$ anticlockwise.

Since $h$ is a linear map,

$$\overrightarrow{CL} = \overrightarrow{CB} + \overrightarrow{BL} = -u + h(\overrightarrow{BJ}) = -u + h(\overrightarrow{AJ}) - h(\overrightarrow{AB}) = -u - v - u = -2u - v$$

and

$$h(\overrightarrow{CG}) = h(\overrightarrow{CD}) + h(\overrightarrow{DG}) = \overrightarrow{DA} + \overrightarrow{DE} = -u - u + v = v - 2u.$$

Therefore

$L$, $C$, $G$ are in line $\iff \overrightarrow{CG} \parallel \overrightarrow{CL} \iff \overrightarrow{CL} \perp h(\overrightarrow{CG})$

$\iff (2u + v)(2u - v) = 0$

$\iff |v| = 2|u|.$

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; L. J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland (a second solution); KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. One incorrect solution was sent in.

The above solutions are related, but the different approaches are interesting.

The problem was taken from a lost sangaku dated 1826, and appears as problem 4.21, page 47, of Fukagawa and Pedoe’s Japanese Temple Geometry Problems. Also given there is the relationship between the sides of the four squares.

* * * * * *
A translate $g$ of a function $f$ is a function $g(x) = f(x + a)$ for some constant $a$. Suppose that one translate of a function $f : \mathbb{R} \to \mathbb{R}$ is odd and another translate is even. Show that $f$ is periodic. Is the converse true?

Solution by the St. Olaf Problem Solving Group, St. Olaf College, Northfield, Minnesota.

We know that there exist $a$ and $b$ such that for every $x$, $f(x + a) = f(-x + a)$ and $f(x + b) = -f(-x + b)$. This implies that $f(2a + x) = f(-x)$ and $f(2b + x) = -f(-x)$. Thus,

$$f(x + 4(a - b)) = f(2a + (x + 2a - 4b)) = f(4b - x - 2a)$$

$$= -f(2a + x - 2b) = -f(2b - x) = f(x).$$

Thus $f(x)$ has period $4(a - b)$. (If $a = b$, then $f(x + a) = f(a - x) = -f(a + x)$ which implies that $f(x)$ is the zero function.)

The converse is not true. The function $f(x) = x - [x]$ provides a counterexample.

Also solved (in about the same way) by SEUNG-JIN BANG, Seoul, Republic of Korea; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; DAVID E. MANES, SUNY at Oneonta, New York; CHRIS WILDHAGEN, Breda, The Netherlands; and the proposers.

* * * * *


Show that

$$\prod_{i=1}^{3} h_i^{-a_i} \leq (3r)^2s,$$

where $a_1, a_2, a_3$ are the sides of a triangle, $h_1, h_2, h_3$ its altitudes, $r$ its inradius, and $s$ its semiperimeter.

I. Solution by Mark Kisin, student, Monash University, Clayton, Australia.

By the generalized A.M.-G.M. inequality,

$$\left(\prod_{i=1}^{3} h_i^{-a_i}\right)^{1/2s} = \left(\prod_{i=1}^{3} h_i^{-a_i}\right)^{-\frac{1}{a_1+a_2+a_3}} \leq \frac{a_1 h_1 + a_2 h_2 + a_3 h_3}{a_1 + a_2 + a_3}.$$

But

$$a_1 h_1 = a_2 h_2 = a_3 h_3 = 2\text{(Area)} = 2rs,$$

so

$$\left(\prod_{i=1}^{3} h_i^{-a_i}\right)^{1/2s} \leq \frac{6rs}{2s} = 3r,$$

and the result follows.
II. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Using the relations \( h_i = 2F/a_i \) (\( i = 1, 2, 3 \)) and \( r = Fs \), where \( F \) is the area of the triangle, the stated inequality becomes

\[
2^{2s} F^{2s} \prod_{i=1}^{3} a_i^{-a_i} \leq 3^{2s} F^{2s} s^{-2s},
\]
i.e.

\[
\prod_{i=1}^{3} a_i^{a_i} \geq \left( \frac{2s}{3} \right)^{2s}.
\]

This inequality is the special case \( n = 3, p_1 = p_2 = p_3 = 1, x_i = a_i \) (\( i = 1, 2, 3 \)) of the following.

THEOREM. Let \( p_1, \ldots, p_n \) be positive real numbers and put \( P_n = p_1 + p_2 + \cdots + p_n \). Then for all positive real numbers \( x_1, \ldots, x_n \),

\[
\prod_{i=1}^{n} x_i^{p_ix_i} \geq \left( \frac{\sum_{i=1}^{n} p_ix_i}{P_n} \right)^{\sum_{i=1}^{n} p_ix_i}.
\]

Proof. The function \( f(x) = x \log x, x > 0 \), is convex. Thus we get

\[
\sum_{i=1}^{n} p_i f(x_i) \geq P_n f \left( \frac{\sum_{i=1}^{n} p_ix_i}{P_n} \right),
\]
i.e., (2). \( \square \)

Since \( f(x) \) is strictly convex, equality holds in (1) if and only if \( a_1 = a_2 = a_3 \).

[Editor’s note. Janous also generalizes the problem to \( n \)-dimensional simplices. Using the method of solution I he obtains

\[
\prod_{i=1}^{n+1} h_i^{F_i} \leq [(n + 1)r]^F,
\]

where \( h_i \) are the altitudes, \( F_i \) the \( (n - 1) \)-dimensional areas of the faces, \( F = \sum F_i \), and \( r \) is the inradius.]

Also solved by L. J. Hut, Groningen, The Netherlands; Murray S. Klamkin, University of Alberta; and Marcin E. Kuczma, Warsaw, Poland.

Klamkin’s solution was very similar to solution II, obtaining a slightly weaker result.

* * * * *


A second-order linear recursive sequence \( \{A_n\}_{n=1}^{\infty} \) is defined by \( A_{n+2} = A_{n+1} + A_n \) for all \( n \geq 1 \), with \( A_1 \) and \( A_2 \) any integers. Select a set \( S \) of any \( 2m \) consecutive elements from this sequence, where \( m \) is an odd integer. Prove that the sum of the numbers in \( S \) is always divisible by the \((m + 2)\)nd element of \( S \), and the multiplying factor is \( L_m \), the \( m \)th Lucas number.
Solution by Chris Wildhagen, Breda, The Netherlands.

Let
\[ \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}. \]

Then \( A_n = r\alpha^n + s\beta^n \) for appropriate constants \( r \) and \( s \). Thus
\[
\sum_{k=n}^{n+2m-1} A_k = \sum_{k=n}^{n+2m-1} (r\alpha^k + s\beta^k) \\
= r\alpha^n \sum_{k=0}^{2m-1} \alpha^k + s\beta^n \sum_{k=0}^{2m-1} \beta^k \\
= r\alpha^n \left( \frac{\alpha^{2m} - 1}{\alpha - 1} \right) + s\beta^n \left( \frac{\beta^{2m} - 1}{\beta - 1} \right) \\
= r\alpha^{n+1}(\alpha^{2m} - 1) + s\beta^{n+1}(\beta^{2m} - 1). \tag{1}
\]

To prove the required result, note that
\[
A_{n+m+1}L_m = (r\alpha^{n+m+1} + s\beta^{n+m+1})(\alpha^{m} + \beta^{m}) \\
= r\alpha^{n+2m+1} + r\alpha^{n+1}(\alpha\beta)^{m} + s\beta^{n+2m+1} + s\beta^{n+1}(\alpha\beta)^{m},
\]
and this equals (1), since \( \alpha\beta = -1 \).

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; MARIA ASCENSiÓN LÓPEZ CHAMORRÓ, I.B. Leopoldo Cano, Valladolid, Spain; BOB PRIELIPP, University of Wisconsin–Oshkosh; KENNETH M. WILKE, Topeka, Kansas; and the proposer.


* * * * *


A parallelogram is called self-diagonal if its sides are proportional to its diagonals. Suppose that \( ABCD \) is a self-diagonal parallelogram in which the bisector of angle \( ADB \) meets \( AB \) at \( E \). Prove that \( AE = AC - AB \).

I. Editor’s comment.

Three readers, MURRAY S. KLAMKIN, University of Alberta, P. PENNING, Delft, The Netherlands, and KENNETH M. WILKE, Topeka, Kansas, point out that this result is false in some cases, namely for self-diagonal parallelograms \( ABCD \) satisfying \( AB/BC = BD/AC \). Here is an example.
It is easy to see that $AB = \sqrt{5}$, $AD = 5$, $BD = \sqrt{10}$, and $AC = 5\sqrt{2}$, so $ABCD$ is self-diagonal. However, it is also pretty obvious that $AE < AC - AB$.

Penning and Wilke then showed that the problem is correct in the other case, that $AB/BC = AC/BD$, which was the case intended by the proposer and considered by all other solvers. Here are two of these solutions.

II. Solution by Jack Garfunkel, Flushing, N.Y.

Denote $AB = a$, $BC = b$, $AC = d_1$, $BD = d_2$, $AE = x$. We are given

$$\frac{a}{b} = \frac{d_1}{d_2},$$

and since $DE$ bisects $\angle ADB$,

$$\frac{x}{a-x} = \frac{b}{d_2},$$

or

$$x = \frac{ab}{b+d_2}.

We therefore have to show that

$$\frac{ab}{b+d_2} = d_1 - a,$$

which simplifies (by (1)) to showing that $2ab = d_1d_2$. Now it is known that

$$d_1^2 + d_2^2 = 2(a^2 + b^2)$$

or, by (1) again,

$$2(a^2 + b^2) = d_1^2 + \frac{b^2d_2^2}{a^2} = \frac{d_1^2}{a^2}(a^2 + b^2).$$

Hence $d_1 = a\sqrt{2}, d_2 = b\sqrt{2}$, and so $d_1d_2 = 2ab$.

III. Solution by Toshio Seimiya, Kawasaki, Japan.

We assume that $AB : AD = AC : BD$, whence we get

$$AB \cdot BD = AD \cdot AC.$$  \hspace{1cm} (2)

We put $\angle ABD = \alpha, \angle DAC = \beta$. Because $ABCD$ is a parallelogram,

$$\text{area } \triangle ABD = \text{area } \triangle ACD.$$

Therefore we get

$$\frac{1}{2}AB \cdot BD \sin \alpha = \frac{1}{2}AD \cdot AC \sin \beta.$$

Hence from (2) we have $\sin \alpha = \sin \beta$, and since $\alpha + \beta < \pi$ we get $\alpha = \beta$. Let $F$ be the point of intersection of $AC$ with $DE$. As

$$\angle AEF = \alpha + \angle EDB = \beta + \angle ADE = \angle AFE,$$
we get $AE = AF$. Because

$$\angle CD F = \angle AEF = \angle AFE = \angle CFD,$$

we have $CF = CD$. Hence

$$AE = AF = AC - CF = AC - CD = AC - AB.$$

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; L. J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANTONIO LUIZ SANTOS, Rio de Janeiro, Brazil; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

* * * * * * *


Two circles $K$ and $K_1$ touch each other externally. The equilateral triangle $ABC$ is inscribed in $K$, and points $A_1, B_1, C_1$ lie on $K_1$ such that $AA_1, BB_1, CC_1$ are tangent to $K_1$. Prove that one of the lengths $AA_1, BB_1, CC_1$ equals the sum of the other two. (The case when the circles are internally tangent was a problem of Florow in Praxis der Mathematik 13, Heft 12, page 327.)

Solution by Marcin E. Kuczma, Warszawa, Poland.

Let $O, O_1, r, r_1$ be the centers and radii of $K$ and $K_1$, and let $T$ be the point of tangency. Assume without loss of generality that $T$ belongs to the shorter arc $AB$. Then by Ptolemaeus, $AT \cdot BC + BT \cdot CA = CT \cdot AB$, whence

$$AT + BT = CT. \quad (1)$$

[This is of course known. —Ed.] Produce $AT$ to cut $K_1$ again in $D$. The isosceles triangles $AOT$ and $TO_1D$ are similar in ratio $r : r_1$, and so $AD/AT = (r + r_1)/r$. Thus

$$AA_1 = \sqrt{AT \cdot AD} = \sqrt{\frac{r + r_1}{r}} \cdot AT,$$

and similarly for $BB_1$ and $CC_1$. The claim results by (1).

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; JEFF HIGHAM, student, University of Toronto; L. J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria;
DAG JONSSON, Uppsala, Sweden; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; HUME SMITH, Chester, Nova Scotia; and the proposer.

Several solvers gave the above solution.


Prove that

\[ 1 + 2 \cos(B + C) \cos(C + A) \cos(A + B) \geq \cos^2(B + C) + \cos^2(C + A) + \cos^2(A + B), \]

where \( A, B, C \) are nonnegative and \( A + B + C \leq \pi \).

I. Solution by C. Festræts-Hamoir, Brussels, Belgium.

Put

\[ B + C = \alpha, \ C + A = \beta, \ A + B = \gamma; \]

then the proposed inequality is \( T \leq 0 \) where

\[ T = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma - 2 \cos \alpha \cos \beta \cos \gamma - 1. \]

Solving \( T = 0 \) by considering \( T \) as a quadratic in \( \cos \alpha \), we obtain

\[ \cos \alpha = \frac{2 \cos \beta \cos \gamma \pm \sqrt{4 \cos^2 \beta \cos^2 \gamma - 4(\cos^2 \beta + \cos^2 \gamma - 1)}}{2} \]

\[ = \cos \beta \cos \gamma \pm \sqrt{(1 - \cos^2 \beta)(1 - \cos^2 \gamma)} \]

\[ = \cos \beta \cos \gamma \pm \sin \beta \sin \gamma = \cos(\beta \mp \gamma). \]

Thus

\[ T = (\cos \alpha - \cos(\beta + \gamma))(\cos \alpha - \cos(\beta - \gamma)) \]

\[ = 4 \sin \frac{\alpha + \beta + \gamma}{2} \sin \frac{\alpha - \beta - \gamma}{2} \sin \frac{\alpha + \beta - \gamma}{2} \sin \frac{\alpha - \beta + \gamma}{2} \]

\[ = 4 \sin(A + B + C) \sin(-A) \sin C \sin B \]

\[ = -4 \sin(A + B + C) \sin A \sin B \sin C \leq 0, \]

since \( A + B + C \leq \pi \) and \( A, B, C \) are non-negative.
II. Solution by the proposer.

Let

\[ x = B + C, \quad y = C + A, \quad z = A + B. \]

Then \( x + y + z \leq 2\pi \). Since \( 2A = y + z - x \), etc., \( x, y, z \) must satisfy the triangle inequality. Hence \( x, y, z \) are the angles of a (possibly degenerate) trihedral angle. If \( \mathbf{P}, \mathbf{Q}, \mathbf{R} \) denote unit vectors along the edges of this trihedral angle from the vertex, the volume \( V \) of the tetrahedron formed from \( \mathbf{P}, \mathbf{Q}, \mathbf{R} \) is given by \( 6V = \mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) \) [p. 26 problem 37 of Spiegel, Vector Analysis (Schaum)]. Squaring, we obtain [p. 33 problem 89 of the same reference]

\[
36V^2 = \begin{vmatrix}
\mathbf{P} \cdot \mathbf{P} & \mathbf{P} \cdot \mathbf{Q} & \mathbf{P} \cdot \mathbf{R} \\
\mathbf{Q} \cdot \mathbf{P} & \mathbf{Q} \cdot \mathbf{Q} & \mathbf{Q} \cdot \mathbf{R} \\
\mathbf{R} \cdot \mathbf{P} & \mathbf{R} \cdot \mathbf{Q} & \mathbf{R} \cdot \mathbf{R}
\end{vmatrix} = \begin{vmatrix}
1 & \cos y & \cos z \\
\cos y & 1 & \cos x \\
\cos z & \cos x & 1
\end{vmatrix}.
\]

On expanding out the determinant, we see that the given inequality corresponds to \( V^2 \geq 0 \). There is equality if and only if \( x + y + z = 2\pi \) or one of \( x, y, z \) equals the sum of the other two; correspondingly, if and only if \( A + B + C = \pi \) or \( ABC = 0 \).

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; and KEE-WAI LAU, Hong Kong.

*   *   *   *   *

JACK GARFUNKEL

One of Crux’s regular contributors since its earliest days, Jack Garfunkel, passed away on New Year’s Eve, 1990. His many proposals in geometry, sometimes very difficult, but often beautiful, were well appreciated by Crux readers. He will be missed. The following information on his life was kindly furnished by his son, Sol Garfunkel.

Jack was born in Poland in 1910 and came to the United States at the age of nine. Although one of the top math majors at City College in New York, he left academic life after graduation to help his family weather the depression. For the next 25 years, he worked manufacturing candy. At the age of 45, he returned to his first love and became a high school mathematics teacher. Over the next 24 years, he taught at Forest Hills High School, supervising over two dozen Westinghouse finalist and semi-finalist winners of the talent search. When he “retired” from high school teaching, he immediately began work as an adjunct professor at Queens College and later at Queensborough Community College, where he taught until November of 1990.

Throughout his two careers as candyman and teacher, he continued to do mathematics, conjecturing and solving problems in synthetic geometry and geometric inequalities. While his formal education ended with his undergraduate work, he never lost his curiosity and love of his subject.
Crux Mathematicorum

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CONTENTS

The Olympiad Corner: No. 123 .................................  R.E. Woodrow  65

Mini-Reviews .........................................................  Andy Liu  74

Problems: 1621-1630 ..................................................  77

Solutions: 693, 1423, 1502, 1504-1512 ...............................  79

Letter to the Editor ..............................................  96
THE OLYMPIAD CORNER

No. 123

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first contest for this issue is the 12th Austrian-Polish Mathematics Competition. Many thanks to Walther Janous, Ursulengymnasium, Innsbruck, Austria, for sending it to me.

12TH AUSTRIAN-POLISH MATHEMATICS COMPETITION

Individual Competition

1st Day—June 28, 1989 (4 1/2 hours)

1. Let \(a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n\) be positive real numbers. Show that
\[
\left( \sum_{k=1}^{n} a_k b_k c_k \right)^3 \leq \left( \sum_{k=1}^{n} a_k^3 \right) \left( \sum_{k=1}^{n} b_k^3 \right) \left( \sum_{k=1}^{n} c_k^3 \right).
\]

2. Each point of the plane (\(R^2\)) is coloured by one of the two colours \(A\) and \(B\). Show that there exists an equilateral triangle with monochromatic vertices.

3. Determine all natural numbers \(N\) (in decimal representation) satisfying the following properties:
   (1) \(N = (aabb)_{10}\), where \((aab)_{10}\) and \((abb)_{10}\) are primes.
   (2) \(N = P_1 \cdot P_2 \cdot P_3\), where \(P_k\) (\(1 \leq k \leq 3\)) is a prime consisting of \(k\) (decimal) digits.

2nd Day—June 29, 1989 (4 1/2 hours)

4. Let \(P\) be a convex polygon in the plane having \(A_1, A_2, \ldots, A_n\) \((n \geq 3)\) as its vertices. Show that there exists a circle containing the entire polygon \(P\) and having at least three adjacent vertices of \(P\) on its boundary.

5. Let \(A\) be a vertex of a cube \(\omega\) circumscribed about a sphere \(\kappa\) of radius 1. We consider lines \(g\) through \(A\) containing at least one point of \(\kappa\). Let \(P\) be the point of \(g \cap \kappa\) having minimal distance from \(A\). Furthermore, \(g \cap \omega\) is \(AQ\). Determine the maximum value of \(AP \cdot AQ\) and characterize the lines \(g\) yielding the maximum.

6. We consider sequences \(\{a_n : n \geq 1\}\) of squares of natural numbers \((> 0)\) such that for each \(n\) the difference \(a_{n+1} - a_n\) is a prime or the square of a prime. Show that all such sequences are finite and determine the longest sequence \(\{a_n : n \geq 1\}\).
Team Competition

June 30, 1989 (4 hours)

7. Functions $f_0, f_1, f_2, \ldots$ are recursively defined by
   (1) $f_0(x) = x$, for $x \in \mathbb{R}$;
   (2) $f_{2k+1}(x) = 3f_{2k}(x)$, where $x \in \mathbb{R}$, $k = 0, 1, 2, \ldots$;
   (3) $f_{2k}(x) = 2f_{2k-1}(x)$, where $x \in \mathbb{R}$, $k = 1, 2, 3, \ldots$.
Determine (with proof) the greater one of the numbers $f_{10}(1)$ and $f_0(2)$.

8. We are given an acute triangle $ABC$. For each point $P$ of the interior or boundary of $ABC$ let $P_a, P_b, P_c$ be the orthogonal projections of $P$ to the sides $a, b$ and $c$, respectively. For such points we define the function

   $$f(P) = \frac{AP_c + BP_a + CP_b}{PP_a + PP_b + PP_c}.$$ 

Show that $f(P)$ is constant if and only if $ABC$ is an equilateral triangle.

9. Determine the smallest odd natural number $N$ such that $N^2$ is the sum of an odd number $(> 1)$ of squares of adjacent natural numbers $(> 0)$.

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The second group of problems are from the other side of the globe, and we thank S. C. Chan of Singapore for forwarding them to us.

SINGAPORE MATHEMATICAL SOCIETY

INTERSCHOOL MATHEMATICAL COMPETITION 1989

Part B, Saturday, 17 June 1989

1. Let $n \geq 5$ be an integer. Show that $n$ is a prime if and only if $n, n_j \neq n_p n_q$ for every partition of $n$ into 4 positive integers, $n = n_1 + n_2 + n_3 + n_4$, and for each permutation $(i, j, p, q)$ of (1, 2, 3, 4).

2. Given arbitrary positive numbers $a$, $b$ and $c$, prove that at least one of the following inequalities is false:

   $$a(1 - b) > \frac{1}{4}, \quad b(1 - c) > \frac{1}{4}, \quad c(1 - a) > \frac{1}{4}.$$ 

3. (a) Show that

   $$\tan \left( \frac{\pi}{12} \right) = \frac{2 - \sqrt{3}}{\sqrt{2 + \sqrt{3}}}.$$ 

   (b) Given any thirteen distinct real numbers, show that there exist at least two, say $x$ and $y$, which satisfy the inequality

   $$0 < \frac{x - y}{1 + xy} < \frac{2 - \sqrt{3}}{\sqrt{2 + \sqrt{3}}}.$$
4. There are $n$ participants in a conference. Suppose (i) every 2 participants who know each other have no common acquaintances; and (ii) every 2 participants who do not know each other have exactly 2 common acquaintances. Show that every participant is acquainted with the same number of people in the conference.

5. In the following diagram, $ABC$ is a triangle, and $X$, $Y$ and $Z$ are respectively the points on the sides $CB$, $CA$ and $BA$ extended such that $XA$, $YB$ and $ZC$ are tangents to the circumcircle of $\triangle ABC$. Show that $X$, $Y$ and $Z$ are collinear.


Having given over last month’s number of the Corner to problems from the ‘archives’, we concentrate this month on solutions to problems given in the 1989 numbers of *Crux*. First an acknowledgement. ‘Also solved’ status should have been given solutions sent in by Michael Selby, University of Windsor, for problems 1, 2 and 7 from the 24th Spanish Olympiad [1989: 67–68], for which we gave solutions in January [1991: 9-10]. His solutions arrived just as the issue was going to press.

When we gave the solutions, in the December 1990 number, to the 1986 Swedish Mathematical Competition, one problem remained unanswered. An alert reader spotted the gap and sent in a solution, which follows.

In the rectangular array

\[
\begin{array}{cccc}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{array}
\]

of \( m \times n \) real numbers, the difference between the maximum and the minimum element in each row is at most \( d \), where \( d > 0 \). Each column is then rearranged in decreasing order so that the maximum element of the column occurs in the first row, and the minimum element occurs in the last row. Show that in the rearranged array the difference between the maximum and the minimum elements in each row is still at most \( d \).

Solution by Andy Liu, University of Alberta.

Suppose to the contrary that in the reorganized array (whose elements we denote \( b_{ij}, 1 \leq i \leq m, 1 \leq j \leq n \)) the difference between the largest element and the smallest element in row \( i \) exceeds \( d \). Let these elements be \( b_{ij} \) and \( b_{ik} \) respectively. Then

\[
b_{1j} \geq b_{2j} \geq \ldots \geq b_{ij} > b_{ik} \geq b_{i+1 k} \geq \ldots \geq b_{mk}.
\]

By the Pigeon Hole Principle, two of these \( n + 1 \) elements must be in the same row in the original array. One of them must be \( b_{pj} \) for some \( p \leq i \) and the other \( b_{qk} \) for some \( q \geq i \). However \( b_{pj} - b_{qk} \geq b_{ij} - b_{ik} > d \). Hence the difference between the largest element and the smallest in that row also exceeds \( d \). This is a contradiction.

* 

In most of the remainder of this column we give the solutions we’ve received to problems of the 1987 Hungarian National Olympiad, given in the April 1989 number of the Corner [1989: 100-101].

1. The surface area and the volume of a cylinder are equal to each other. Determine the radius and the altitude of the cylinder if both values are even integers.

Solution by Bob Prielipp, University of Wisconsin–Oshkosh, and by Michael Selby, University of Windsor.

Let \( r \) be the radius and \( h \) be the altitude of the given cylinder. Then \( 2\pi rh + 2\pi r^2 \) is the surface area of the cylinder and \( \pi r^2h \) is its volume. Since the surface area and the volume of the cylinder are equal, \( 2\pi rh + 2\pi r^2 = \pi r^2h \) so \( (r-2)(h-2)=4 \). Because \( r \) and \( h \) are both even positive integers, \( r-2=2 \) and \( h-2=2 \). Thus \( r=4 \) and \( h=4 \).

2. Cut the regular (equilateral) triangle \( AXY \) from rectangle \( ABCD \) in such a way that the vertex \( X \) is on side \( BC \) and the vertex \( Y \) is on side \( CD \). Prove that among the three remaining right triangles there are two, the sum of whose areas equals the area of the third.
Solution by Michael Selby, University of Windsor, and by D. J. Smeenk, Zaltbommel, The Netherlands.

First, such a triangle is not possible in every rectangle. In fact a necessary condition is that the sides \(a \leq b\) of the rectangle must satisfy \(a \geq b\sqrt{3}/2\).

Let us assume such a triangle is possible. Let \(s\) be the length of the side of the equilateral triangle. We claim that \([XYC] = [ABX] + [ADY]\), where \([T]\) denotes the area of triangle \(T\). Indeed

\[
[ABX] = \frac{1}{2} s^2 \cos \theta_1 \sin \theta_1 = \frac{1}{4} s^2 \sin 2\theta_1
\]

and

\[
[ADY] = \frac{1}{2} s^2 \cos \theta_2 \sin \theta_2 = \frac{1}{4} s^2 \sin 2\theta_2 = \frac{1}{4} s^2 \sin \left(\frac{\pi}{3} - 2\theta_1\right)
\]

\[
= \frac{1}{4} s^2 \sqrt{3} \cos 2\theta_1 - \frac{1}{8} s^2 \sin 2\theta_1,
\]

thus

\[
[ABX] + [ADY] = \frac{1}{8} s^2 \sin 2\theta_1 + \frac{\sqrt{3}}{8} s^2 \cos 2\theta_1.
\]

Now for \(\triangle XYC\), \(\angle YXC = \pi/6 + \theta_1\). Therefore,

\[
[XYC] = \frac{s^2}{4} \sin \left(\frac{\pi}{3} + 2\theta_1\right)
\]

\[
= \frac{s^2 \sqrt{3}}{4} \cos 2\theta_1 + \frac{s^2}{4} \frac{1}{2} \sin 2\theta_1.
\]

This is exactly (1), the sum of the areas.

3. Determine the minimum of the function

\[
f(x) = \sqrt{a^2 + x^2} + \sqrt{(b - x)^2 + c^2}
\]

where \(a, b, c\) are positive numbers.

Solution by Mangho Ahuja, Southeast Missouri State University, and by D. J. Smeenk, Zaltbommel, The Netherlands.

Let \(AC = a\), \(AB = b\) and \(BD = c\). Let \(P\) be a point on \(AB\) and let \(x = AP\), so that \(BP = b - x\). Then \(f(x) = CP + PD\). To minimize \(CP + PD\), we follow the method of reflection (see Z. A. Melzak, Companion to Mathematics, John Wiley & Sons, 1973, pp. 26-27). Let \(D'\) be the reflection of \(D\) in the line \(AB\). Since triangles \(PBD\) and \(PBD'\) are congruent, \(PD = PD'\) and \(f(x) = CP + PD'\). As \(x\) varies, \(P\) changes its position. But the distance \(CP + PD'\) will be a minimum when \(P\) lies on the line \(CD'\). The minimum value of \(f(x)\) is then \(CP + PD' = CD'\). Let \(CL\) be the perpendicular...
from $C$ to the line $DD'$. From the right triangle $CLD'$,

$$CD' = \sqrt{(LD)^2 + (CL)^2} = \sqrt{(a + c)^2 + b^2}.$$

[Editor’s note. A solution via the calculus was submitted by Michael Selby, University of Windsor.]

4. Consider points $A$ and $B$ on given rays (semilines) starting from $C$, such that the sum $CA + CB$ is a given constant. Show that there is a point $D \neq C$ such that for each position of $A$ and $B$ the circumcircle of triangle $ABC$ passes through $D$.  

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

Let $A$ and $B$ be given and let $D$, be the circumcircle of $\triangle ABC$. The interior bisector of $\angle ACB$ intersects $AB$ for the second time in the required point $D$. To see this, consult the figure. If $CA' + CB' = CA + CB$ then $AA' = BB'$ (1). Also $DA = DB$ (2) since $\angle ABD = \angle ACD = \angle DCB = \angle DAB$. Assume without loss that $CB \geq CA$. Quadrilateral $DBCA$ is inscribed in a circle, hence $\angle DAA' = \angle DBB'$ (3). From (1), (2) and (3) $\triangle DAA' \cong \triangle DBB'$. Thus $\angle ADA' = \angle BDB'$ and $\angle A'BD = \angle ADB$. Thus $D$ is a point on the circumcircle of $\triangle A'BD'C$.

6. $N$ is a 4-digit perfect square all of whose decimal digits are less than seven. Increasing each digit by three we obtain a perfect square again. Find $N$.

Solutions by Stewart Metchette, Culver City, California; John Morvay, Springfield, Missouri; Bob Prielipp, University of Wisconsin–Oshkosh; D.J. Smeenk, Zaltbommel, The Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $N = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$, where $a$, $b$, $c$ and $d$ are integers such that $1 \leq a \leq 6$ and $0 \leq b, c, d \leq 6$. By hypothesis $N = n^2$ for some positive integer $n$. Hence $n^2 = N \leq 6666$ so $n \leq 81$. Also by hypothesis

$$(a + 3) \cdot 10^3 + (b + 3) \cdot 10^2 + (c + 3) \cdot 10 + (d + 3) = m^2$$

for some positive integer $m$. Thus $m^2 - n^2 = 3333$, making $(m + n)(m - n) = 3 \cdot 11 \cdot 101$. Because $m + n > m - n$ and $n \leq 81$, it follows that $m + n = 101$ and $m - n = 33$, so $n = 34$. Therefore $N = 1156$.

7. Let $a$, $b$, $c$ be the sides and $\alpha$, $\beta$, $\gamma$ be the opposite angles of a triangle. Show that if

$$ab^2 \cos \alpha = bc^2 \cos \beta = ca^2 \cos \gamma$$

then the triangle is equilateral.
From the equality $ab^2 \cos \alpha = bc^2 \cos \beta$ and the law of cosines, we get

$$ab \left( \frac{b^2 + c^2 - a^2}{2bc} \right) = c^2 \left( \frac{a^2 + c^2 - b^2}{2ac} \right),$$
equivalently

$$a^2(b^2 + c^2 - a^2) = c^2(a^2 + c^2 - b^2)$$
and

$$a^2b^2 + b^2c^2 = a^4 + c^4. \quad (1)$$

Similarly we obtain

$$b^2c^2 + a^2c^2 = a^4 + b^4 \quad (2)$$
and

$$a^2b^2 + a^2c^2 = b^4 + c^4. \quad (3)$$
Adding equations (1), (2) and (3) we have

$$2a^2b^2 + 2a^2c^2 + 2b^2c^2 = 2a^4 + 2b^4 + 2c^4.$$Equivalently

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 = 0,$$so $a^2 = b^2 = c^2$ and $a = b = c.$

8. Let $u$ and $v$ be two real numbers such that $u$, $v$ and $uv$ are roots of a cubic polynomial with rational coefficients. Prove or disprove that $uv$ is rational.

Solutions by Bob Prielipp, University of Wisconsin–Oshkosh; Michael Selby, University of Windsor; D.J. Smeenk, Zaltbommel, The Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We disprove.

Clearly we may assume, without any loss of generality, that $u$, $v$ and $uv$ are roots of a monic cubic polynomial

$$P(x) = x^3 + ax^2 + bx + c$$
where $a$, $b$ and $c$ are rationals. Since $P(x) = (x - u)(x - v)(x - uv)$ we have

$$u + v + uv = -a,$$
$$uv + uv(u + v) = b,$$
$$a^2v^2 = -c.$$From this we have

$$auv = -uv(u + v) - u^2v^2 = uv - b + c.$$
or \((a - 1)uw = c - b\). Thus \(uv\) is rational if \(a \neq 1\). However, if \(a = 1\), then \(uv\) need not be rational since in this case \(b = c\) and

\[
P(x) = x^3 + x^2 + cx + c = (x + 1)(x^2 + c)
\]

and thus \(\{u, v, uv\} = \{-1, \sqrt{-c}, -\sqrt{-c}\}\) which are real for \(c \leq 0\). Thus if we take for example \(c = -2\), then \(u = -1, v = \sqrt{2}\) and \(uv = -\sqrt{2}\) are roots of the cubic polynomial \(x^3 + x^2 - 2x - 2\).

9. The lengths of the sides of a triangle are 3, 4 and 5. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

Editor's note. An equivalent problem appeared as problem 1 of the 1985 Canadian Mathematics Olympiad (the answer, “one”, was given in the problem). A solution is given on [1985: 272]. A solution to the present problem was sent in by Michael Selby, University of Windsor.

11. The domain of function \(f\) is \([0, 1]\), and for any \(x_1 \neq x_2\)

\[
|f(x_1) - f(x_2)| < |x_1 - x_2|.
\]

Moreover, \(f(0) = f(1) = 0\). Prove that for any \(x_1, x_2\) in \([0, 1]\),

\[
|f(x_1) - f(x_2)| < \frac{1}{2}.
\]

Solutions by Bob Prielipp, University of Wisconsin–Oshkosh, and by Michael Selby, University of Windsor.

First \(|f(x) - f(0)| \leq |x - 0|\), i.e. \(|f(x)| \leq x\) and the inequality is strict for \(x \neq 0\). Also \(|f(x) - f(1)| \leq |x - 1|\), i.e. \(|f(x)| \leq 1 - x\) with strict inequality for \(x \neq 1\). Therefore

\[
|f(x)| \leq \min(x, 1 - x),
\]

and the inequality is strict unless \(x = 0\) or \(x = 1\). Let \(x_1, x_2 \in [0, 1]\). If \(|x_1 - x_2| \leq 1/2\), then

\[
|f(x_1) - f(x_2)| \leq |x_1 - x_2| \leq 1/2.
\]

This gives \(|f(x_1) - f(x_2)| < 1/2\) since \(\ast\) is strict unless \(x_1 = x_2\) and this case is trivial. So suppose \(|x_1 - x_2| > 1/2\). Without loss of generality suppose that \(x_1 \in (1/2, 1]\) and \(x_2 \in [0, 1/2)\). Then

\[
|f(x_1) - f(x_2)| \leq |f(x_1)| + |f(x_2)| \leq 1 - x_1 + x_2 = 1 - (x_1 - x_2) < \frac{1}{2}.
\]

Therefore, in all cases \(|f(x_1) - f(x_2)| < 1/2\) for all \(x_1 \neq x_2\).

* 

The last two solutions we present this number are to problems from the May 1989 column.
Determine the sum of all the divisors $d$ of $N = 19^{88} - 1$ which are of the form $d = 2^a \cdot 3^b$ with $a, b > 0$.

Solution adapted by the editors from one by John Morvay, Springfield, Missouri.

Note that

$$19^{88} - 1 = (19^{11} - 1)(19^{11} + 1)(19^{22} + 1)(19^{44} + 1).$$

We look for integers $m, n$ such that $2^m 3^n | 19^{88} - 1$, where $m, n$ are the greatest such integers. It is clear from the above factorization that $n = 2$, as $3^2 | 19^{11} - 1$ while $3^3$ does not divide into $19^{11} - 1$, and the other factors are $\equiv -1 \mod 3$. Also $m = 5$, since $19^{11} + 1$ is divisible by 4 but not 8, and the other three factors are divisible by only 2. From this, the required sum of the divisors is $(3 + 9)(2 + 4 + 8 + 16 + 32) = 744$.

* 

Let $a_1, \ldots, a_{1988}$ be positive real numbers whose arithmetic mean equals 1988. Show

$$\sqrt[1988]{\prod_{i=1}^{1988} \prod_{j=1}^{1988} (1 + \frac{a_i}{a_j})} \geq 2^{1988}$$

and determine when equality holds.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We shall prove the following general result which shows that the assumption about the arithmetic mean of the $a_k$’s is redundant.

**Theorem:** If $a_i > 0$ for all $i = 1, 2, \ldots, n$ then

$$\left( \prod_{i=1}^{n} \prod_{j=1}^{n} (1 + \frac{a_i}{a_j}) \right)^{1/n} \geq 2^n$$

with equality if and only if the $a_k$’s are equal.

**Proof.** It is well known (cf [1], [2]) and easy to show that if $a_1 a_2 \cdots a_n = b^n$, then

$$\prod_{i=1}^{n} (1 + a_i) \geq (1 + b)^n$$

with equality if and only if the $a_k$’s are equal. Using this, we obtain for each fixed $i$,

$$\prod_{j=1}^{n} (1 + \frac{a_i}{a_j}) \geq (1 + \frac{a_i}{b})^n$$

and hence

$$\left( \prod_{i=1}^{n} \prod_{j=1}^{n} (1 + \frac{a_i}{a_j}) \right)^{1/n} \geq \prod_{i=1}^{n} (1 + \frac{a_i}{b}) \geq (1 + \frac{b}{b})^n = 2^n$$
with equality if and only if the $a_k$’s are equal.

References:

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The Olympiad season is upon us. Send me your national and regional contests, as well as your nice solutions to problems from the Corner.

* * *

MINI-REVIEWS
by ANDY LIU

INDIVIDUAL TITLES


This is the second edition of the first of five books by an outstanding scientist and educator on his theory and methods of problem-solving. Here, numerous examples illustrate the Pólya method which divides the task into four phases: understanding the problem, devising a plan, carrying out the plan and looking back.


The subtitle of this volume is “Induction and Analogy in Mathematics”. It is a continuation and elaboration of the ideas propounded in “How To Solve It”. It discusses inductive reasoning and making conjectures, with examples mainly from number theory and geometry. The transition to deductive reasoning is via mathematical induction.


The subtitle of this volume is “Patterns of Plausible Inference”. It is primarily concerned with the role of plausible reasoning in the discovery of mathematical facts. Two chapters in probability provide most of the illustrative examples.


This volume contains part one of the work, titled “Patterns”, and the first two chapters of part two, titled “Toward a General Method”. The first part gives practices for pattern recognition in geometric loci, analytic geometry, recurrence relations and interpolation. The two chapters in the second part discuss general philosophy in problem-solving.


This volume contains the remaining nine chapters of part two of the work. More specific advice on the art and science of problem-solving is offered. There is a chapter on learning, teaching and learning teaching.
This is a do-it-yourself package through which the reader can learn and develop methods of problem-solving. The first part of the book contains four short investigations, and the second part two extended ones. Each investigation is conducted via a structured sequence of questions.

This book contains thirty-one chapters. The first twenty-nine are groups of related puzzles. The chapters are independent, except for three on counting and three on primes. In each chapter, commentaries follow the statements of the puzzles. Answers are given at the end of the book. Chapter 30 reexamines four of the earlier puzzles while Chapter 31 discusses the role of puzzles in mathematics.

This excellent book contains three hundred and fifty problems in arithmetic and algebra, many from papers of the U.S.S.R. Olympiads. It is the first of three volumes, but unfortunately, the volumes on plane geometry and solid geometry have not been translated into English.

All the Best from the Australian Mathematics Competition, edited by J.D. Edwards, D.J. King and P.J. O’Halloran, Australian Mathematics Competition, 1986. (paperback, 220 pp.)
This book contains four hundred and sixty-three multiple-choice questions taken from one of the world’s most successful mathematics competitions. They are grouped by subject area to facilitate the study of specific topics.

This booklet contains the questions and solutions of the first ten Canadian Mathematics Olympiads. Each of the first five papers consists of ten questions. The numbers of questions in the remaining five vary between six and eight.

This booklet contains the questions and solutions of the Canadian Mathematics Olympiads from 1979 to 1985. Each paper consists of five questions.

To date, half of this work has appeared in a preliminary version in the form of five booklets. In addition to problems and solutions, a mathematical “tool chest” is appended to each booklet.

This is the definitive treatise on mathematical games. As the subtitle “Games in General” suggests, the general theory of mathematical games is presented in this first volume, but there are also plenty of specific games to be analysed, played and enjoyed. The book is written with a great sense of humor, and is profusely illustrated, often in bright colours.


The subtitle of this volume is “Games in Particular”. Here, all kinds of mathematical games, classical as well as brand new, are presented attractively. Most of them are two-player games. There are two chapters devoted to one-player games or solitaire puzzles, and the book concludes with a chapter on a zero-player game!


Jerry Slocum has probably the largest puzzle collection in the world. This book features a small subset of his mechanical puzzles, that is, puzzles made of solid pieces that must be manipulated by hand to obtain a solution. They are classified into ten broad categories, with enough information to make most of them and to solve some of them. The book is full of striking full-colour plates.


This book contains thirty articles dedicated to Martin Gardner for his sixty-fifth birthday. They reflect part of his mathematical interest, and are classified under the headings Games, Geometry, Two-Dimensional Tiling, Three-Dimensional Tiling, Fun and Problems, and Numbers and Coding Theory.

Mathematical Snapshots, by H. Steinhaus, Oxford University Press, 1983. (paperback, 311 pp.)

This is an outstanding book on significant mathematics presented in puzzle form. Topics include dissection theory, the golden ratio, numeration systems, tessellations, geodesics, projective geometry, polyhedra, Platonic solids, mathematical cartography, spirals, ruled surfaces, graph theory and statistics.

Mathematics Can Be Fun, by Y.I. Perelman, Mir Publishers, 1979. (hardcover, 400 pp.)

This is a translation of two books in Russian, “Figures for Fun” and “Algebra Can Be Fun”. The former is an excellent collection of simple puzzles. The latter is a general discourse of algebra with quite a few digressions into number theory.

Fun with Maths and Physics, by Y.I. Perelman, Mir Publishers, 1984. (hardcover, 374 pp.)

The first half of this beautiful book describes a large number of interesting experiments in physics. The second half consists of a large collection of mathematical puzzles.
The Moscow Puzzles, by B.A. Kordemsky, Charles Scribner’s Sons, 1972. (paperback, 309 pp.)
This is the translation of the outstanding single-volume puzzle collection in the history of Soviet mathematics. Many of the three hundred and fifty-nine problems are presented in amusing and charming story form, often with illustrations.

The Tokyo Puzzles, by K. Fujimura, Charles Scribner’s Sons, 1978. (paperback, 184 pp.)
This is the translation of one of many books from the leading puzzlist of modern-day Japan. It contains ninety-eight problems, most of them previously unfamiliar to the western world.

536 Puzzles and Curious Problems, by H.E. Dudeney, Charles Scribner’s Sons, 1967. (paperback, 428 pp.)
This book is a combination of two out-of-print works of the author, “Modern Puzzles” and “Puzzles and Curious Problems”. Together with Dover’s “Amusements in Mathematics”, they constitute a substantial portion of Dudeney’s mathematical problems. Those in this book are classified under three broad headings, arithmetic and algebra, geometry, and combinatorics and topology.

Science Fiction Puzzle Tales, by Martin Gardner, Clarkson N. Potter, 1981. (paperback, 148 pp.)
This is the first of three collections of Martin Gardner’s contribution to Isaac Asimov’s Science Fiction Magazine. The book contains thirty-six mathematical puzzles in science fiction settings. When solutions are presented, related questions are often raised.

This is the sequel to “Science Fiction Tales” and the predecessor of New Mathematical Library’s “Riddles of the Sphinx”. It contains thirty-seven mathematical puzzles plus further questions raised in the answer sections.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.
To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **October 1, 1991**, although solutions received after that date will also be considered until the time when a solution is published.

**1621*. Proposed by Murray S. Klamkin, University of Alberta.** (Dedicated to Jack Garfinkel.)

Let \( P \) be a point within or on an equilateral triangle and let \( c_1 \leq c_2 \leq c_3 \) be the lengths of the three concurrent cevians through \( P \). Determine the minimum value of \( c_2/c_3 \) over all \( P \).

**1622. Proposed by Marcin E. Kuczma, Warszawa, Poland.**

Let \( n \) be a positive integer.

(a) Prove the inequality

\[
\frac{a^{2n} + b^{2n}}{2} \leq \left( \left( \frac{a + b}{2} \right)^2 + (2n - 1) \left( \frac{a - b}{2} \right)^2 \right)^n
\]

for real \( a, b \), and find conditions for equality.

(b) Show that the constant \( 2n - 1 \) in the right-hand expression is the best possible, in the sense that on replacing it by a smaller one we get an inequality which fails to hold for some \( a, b \).

**1623. Proposed by Stanley Rabinowitz, Westford, Massachusetts.** (Dedicated to Jack Garfinkel.)

Let \( \ell \) be any line through vertex \( A \) of triangle \( ABC \) that is external to the triangle. Two circles with radii \( r_1 \) and \( r_2 \) are each external to the triangle and each tangent to \( \ell \) and to line \( BC \), and are respectively tangent to \( AB \) and \( AC \).

(a) If \( AB = AC \), prove that as \( \ell \) varies, \( r_1 + r_2 \) remains constant and equal to the height of \( A \) above \( BC \).

(b) If \( \Delta ABC \) is arbitrary, find constants \( k_1 \) and \( k_2 \), depending only on the triangle, so that \( k_1 r_1 + k_2 r_2 \) remains constant as \( \ell \) varies.

**1624*. Proposed by Walther Janous, Ursulineninngymnasium, Innsbruck, Austria.**

Let \( [n] = \{1, 2, \ldots, n\} \). Choose independently and at random two subsets \( A, B \) of \( [n] \). Find the expected size of \( A \cap B \). What if \( A \) and \( B \) must be different subsets?

**1625. Proposed by Toshio Seimiya, Kawasaki, Japan.**

Isosceles triangles \( A_3A_1O_2 \) and \( A_1A_2O_3 \) (with \( O_2A_1 = O_2A_3 \) and \( O_3A_1 = O_3A_2 \)) are described on the sides \( A_3A_1 \) and \( A_1A_2 \) outside the triangle \( A_1A_2A_3 \). Point \( O_1 \) outside \( \Delta A_1A_2A_3 \) is such that \( \angle O_1A_2A_3 = \frac{1}{2} \angle A_1O_2A_3 \) and \( \angle O_1A_3A_2 = \frac{1}{2} \angle A_1O_3A_2 \). Prove that \( A_1O_1 \perp O_2O_3 \), and that

\[
\overrightarrow{A_1O_1} : \overrightarrow{O_2O_3} = 2 \overrightarrow{O_1T} : \overrightarrow{A_2A_3},
\]

where \( T \) is the foot of the perpendicular from \( O_1 \) to \( A_2A_3 \).
Determine the average number of throws of a standard die required to obtain each face of the die at least once.

1627. Proposed by George Tsintsifas, Thessaloniki, Greece. (Dedicated to Jack Garfunkel.)
Two perpendicular chords $MN$ and $ET$ partition the circle $(O, R)$ into four parts $Q_1, Q_2, Q_3, Q_4$. We denote by $(O_i, r_i)$ the incircle of $Q_i$, $1 \leq i \leq 4$. Prove that
$$r_1 + r_2 + r_3 + r_4 \leq 4(\sqrt{2} - 1)R.$$

1628*. Proposed by Remy van de Ven, student, University of Sydney, Sydney, Australia.
Prove that
$$(1 - r)^k \sum_{i=1}^{\infty} \binom{k + i - 1}{i} \frac{r^i}{i^2 \left(\frac{1}{(k + i - 1)^2} + \cdots + \frac{1}{(k + 1)^2}\right)} = \sum_{i=1}^{\infty} \frac{r^i}{i^2 \left(\frac{1}{(k + i - 1)^2}\right)},$$
where $k$ is a positive integer.

1629. Proposed by Rossen Ivanov, student, St. Kliment Ohridsky University, Sofia, Bulgaria.
In a tetrahedron $x$ and $v$, $y$ and $u$, $z$ and $t$ are pairs of opposite edges, and the distances between the midpoints of each pair are respectively $l$, $m$, $n$. The tetrahedron has surface area $S$, circumradius $R$, and inradius $r$. Prove that, for any real number $a$ with $0 \leq a \leq 1$,
$$x^{2a}v^{2a}l^2 + y^{2a}u^{2a}m^2 + z^{2a}t^{2a}n^2 \geq \left(\frac{\sqrt{3}}{4}\right)^{1-a} \left(2S\right)^{1+a}(Rr)^a.$$

1630. Proposed by Isao Ashiba, Tokyo, Japan.
Maximize
$$a_1 a_2 + a_3 a_4 + \cdots + a_{2n-1} a_{2n}$$
over all permutations $a_1, a_2, \ldots, a_{2n}$ of the set $\{1, 2, \ldots, 2n\}$.

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

On a $4 \times 4$ tick-tack-toe board, a winning path consists of four squares in a row, column, or diagonal. In how many ways can three X's be placed on the board, not all on the same winning path, so that if a game is played on this partly-filled board, X going first, then X can absolutely force a win?
Solution by Sam Maltby, student, University of Calgary.

There are \( \binom{16}{3} = 560 \) possible arrangements for the first three X’s, but 40 of these have them on a winning path (four possibilities for each of the four rows, four columns and two diagonals). The remaining 520 cases may be reduced to the following 71, by eliminating equivalent boards obtained by rotations and reflections.
These may be further reduced to 25 by applying the following two operations:

(i) interchanging the second and third columns and then the second and third rows, i.e.,

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]

\[
\rightarrow
\begin{array}{cccc}
1 & 3 & 2 & 4 \\
9 & 11 & 10 & 12 \\
5 & 7 & 6 & 8 \\
13 & 15 & 14 & 16 \\
\end{array}
\]

(ii) the “inversion”

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]

\[
\rightarrow
\begin{array}{cccc}
6 & 5 & 8 & 7 \\
2 & 1 & 4 & 3 \\
14 & 13 & 16 & 15 \\
10 & 9 & 12 & 11 \\
\end{array}
\]

It is easy to verify that each of these operations leaves the set of ten winning rows, columns and diagonals fixed. Therefore if there is a winning strategy for X on a certain board, there will also be a winning strategy on any board resulting from these operations. Thus we need only examine one board from each equivalence class determined by these operations. There are 25 such equivalence classes, namely:

\[
\begin{align*}
\{1,16,49,70\} & \quad \{2,18\} & \quad \{3,17,31,36\} & \quad \{4,19,56,67\} & \quad \{5,55\} \\
\{6,14\} & \quad \{7,13,29,42\} & \quad \{8,15,50,58\} & \quad \{9,20,54,69\} & \quad \{10,22,45,47\} \\
\{11,21,59,65\} & \quad \{12,23,57,61\} & \quad \{24,25,28,33\} & \quad \{26,71\} & \quad \{27,62\} \\
\{30,38\} & \quad \{32,37\} & \quad \{34,35\} & \quad \{39,64\} & \quad \{40,43,52,63\} \\
\{41,60\} & \quad \{44,46,51,68\} & \quad \{48\} & \quad \{53\} & \quad \{66\} \\
\end{align*}
\]

The representatives which we use are given in bold print.

For eleven of these boards, X can force a win by determining all of O’s moves (i.e., making three X’s in a line so that O must take the fourth), and ending up with two lines with three X’s each so that O cannot block them both. In the following figure, as well as the original three X’s on each of these eleven boards we also show X’s successive moves by numbers 1,2,3 and so on. Also shown (in parentheses) is the number of the original 520 boards equivalent to each board.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 3 & 2 & 4 \\
9 & 11 & 10 & 12 \\
5 & 7 & 6 & 8 \\
13 & 15 & 14 & 16 \\
\end{array}
\]

\[
\begin{array}{cccc}
6 & 5 & 8 & 7 \\
2 & 1 & 4 & 3 \\
14 & 13 & 16 & 15 \\
10 & 9 & 12 & 11 \\
\end{array}
\]

The representatives which we use are given in bold print.

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For six other cases, X has a winning strategy, but he cannot dictate O’s moves. The positions in the following table are given by coordinates as follows:

<p>| | | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>11</td>
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<td>14</td>
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<tr>
<td>41</td>
<td>42</td>
<td>43</td>
<td>44</td>
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</table>

Also given is X’s first move.

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<tbody>
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<td>X</td>
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</tr>
</tbody>
</table>

**If O takes**

- 33,34,43 or 44
- 11,12,21 or 22
- 13,14,23 or 24

**X takes**

- 11,12 and 14
- 14,34 and 44
- 11,22 and 33

This leaves eight classes:

- 10 (32)
- 11 (32)
- 39 (16)
- 40 (16)
- 44 (32)
- 48 (8)
- 53 (8)
- 66 (8)

Having been unable to find a winning strategy for any of them and having convinced myself that there were none, I put the problem to my Atari 1040ST computer. After about 5 hours, it agreed that O can always force a tie in these situations. Therefore X has a winning strategy in 368 of the 520 starting positions.

* * * * *

Given positive integers $k$, $m$, $n$, find a polynomial $p(x)$ with real coefficients such that

$$(x - 1)^n \left[p(x)\right]^m - x^k.$$ 

What is the least possible degree of $p$ (in terms of $k$, $m$, $n$)?

II. Comment by Rex Westbrook, University of Calgary

This is in response to the editor’s request for a proof that the published answer

$$1 + \sum_{j=1}^{n-1} \frac{k_m(k_m - 1) \cdots (k_m - j + 1)}{j!} (x - 1)^j$$

can be written

$$\frac{k_m(k_m - 1) \cdots (k_m - n + 1)}{(n - 1)!} \sum_{i=0}^{n-1} \frac{(-1)^{n-1-i}}{k_m - i} \binom{n - 1}{i} x^i.$$ 

Set the former expression equal to $f_n(x)$ and the latter equal to $g_n(x)$. The asked equality obviously holds if $k/m = t \leq n - 1$, $t$ an integer, because then both expressions reduce to $x^t$. Otherwise, consider

\[
\frac{d}{dx}[x^{-k/m}f_n(x)] = -\frac{k}{m}x^{-\left(\frac{k}{m}+1\right)}
+ \sum_{j=1}^{n-1} \frac{k_m(k_m - 1) \cdots (k_m - j + 1)}{j!} \left[ j(x - 1)^{j-1}x^{-k/m} - \frac{k_m(x - 1)^{j-1}}{x^{k/m}} \right]
= x^{-\left(\frac{k}{m}+1\right)} \left[ -\frac{k}{m} + \sum_{j=1}^{n-1} \frac{k_m(k_m - 1) \cdots (k_m - j + 1)}{j!} (x - 1)^{j-1} \left( jx - \frac{k_m(x - 1)^{j-1}}{x^{k/m}} \right) \right]
= x^{-\left(\frac{k}{m}+1\right)} \left[ -\frac{k}{m} + \sum_{j=1}^{n-1} \frac{k_m(k_m - 1) \cdots (k_m - j + 1)}{(j-1)!} (x - 1)^{j-1}
- \sum_{j=1}^{n-1} \frac{k_m \cdots (k_m - j + 1)(k_m - j)}{j!} (x - 1)^j \right]
= x^{-\left(\frac{k}{m}+1\right)} \left[ -\frac{k_m(k_m - 1) \cdots (k_m - (n - 1))}{(n - 1)!} (x - 1)^{n-1} \right].
\]

and

\[
\frac{d}{dx}[x^{-k/m}g_n(x)] = \frac{k_m(k_m - 1) \cdots (k_m - (n - 1))}{(n - 1)!} \frac{d}{dx} \left[ \sum_{i=0}^{n-1} \frac{(-1)^{n-1-i}}{k_m - i} \binom{n - 1}{i} x^i \right].
\]
\[
= x^{-(\frac{k}{m}+1)} \left[ \left( \frac{k}{m} \right) - 1 \cdots \left( \frac{k}{m} - (n-1) \right) \right] (x-1)^{n-1}.
\]

Therefore
\[
\frac{d}{dx} \left[ x^{-k/m} (g_n(x) - f_n(x)) \right] = 0,
\]
so
\[
x^{-k/m} (g_n(x) - f_n(x)) = \text{constant}.
\]

If either \( k \neq m \) is not an integer, or is an integer \( \geq n \), then since \( g_n = f_n \), the above constant must be 0.


\( AB \) is a chord, not a diameter, of a circle with centre \( O \). The smaller arc \( AB \) is divided into three equal arcs \( AC, CD, DB \). Chord \( AB \) is also divided into three equal segments \( AC', C'D', D'B \). Let \( CC' \) and \( DD' \) intersect in \( P \). Show that \( \angle APB = \frac{1}{3} \angle AOB \).

Solution by Mark Kisin, student, Monash University, Clayton, Australia.

Let
\[ A' = PA \cap CD, \quad B' = PB \cap CD. \]

Then
\[ \Delta P A C' \sim \Delta P A' C \]
and
\[ \Delta P C' D' \sim \Delta P C D, \]
so that
\[ \frac{AC'}{AC} = \frac{PC'}{PC} = \frac{C'D'}{CD}. \]

But \( AC' = C'D' \), so \( A'C = CD = AC \). Therefore
\[ \angle BAC = \angle ACA' = \pi - \angle CA'A = \angle CA'A = \pi - 2 \angle CA'A \]
\[ = \pi - \angle CA'A - \angle BB'D = \angle APB. \]

But \( C \) and \( D \) trisect the arc \( AB \), so
\[ \angle APB = \angle BAC = \frac{1}{2} \angle BOC = \frac{1}{2} \cdot \frac{2}{3} \angle AOB = \frac{1}{3} \angle AOB. \]

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; JEFF HIGHAM, student, University of Toronto; L. J. HUT,
Let $A_1, A_2, \ldots, A_n$ be a circumscribable $n$-gon with incircle of radius 1, and let $F_1, F_2, \ldots, F_n$ be the areas of the $n$ corner regions inside the $n$-gon and outside the incircle. Show that
\[
\frac{1}{F_1} + \cdots + \frac{1}{F_n} \geq \frac{n^2}{n \tan(\pi/n) - \pi}.
\]
Equality holds for the regular $n$-gon.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Write $\angle A_{i-1}OA_i = 2x_i$ ($A_0 = A_n$), where $O$ is the incenter. Then $\sum x_i = \pi$, $0 < x_i < \pi/2$ and $F_i = \tan x_i - x_i$. Applying Jensen’s inequality
\[
\sum f(x_i) \geq n f(\sum x_i/n)
\]
to the (strictly) convex function
\[
f(x) = \frac{1}{\tan x - x},
\]
we get the claim. The convexity of $f$ is justified, for example, by
\[
f'(x) = -(1 - x \cot x)^{-2}.
\]
As $x$ grows from 0 to $\pi/2$, $x \cot x$ falls, hence $f'(x)$ grows.
Let $x_1 = 1$ and
\[ x_{n+1} = \frac{1}{x_n} \left( \sqrt{1 + x_n^2} - 1 \right). \]

Show that the sequence $(2^n x_n)$ converges and find its limit.

Solution by Jeff Higham, student, University of Toronto.

We first show by induction on $n$ that
\[ x_n = \tan\left(\frac{\pi}{2^{n+1}}\right) \tag{1} \]
for all $n \in \mathbb{N}$. Since $x_1 = 1 = \tan(\pi/4)$, (1) is true for $n = 1$. Suppose (1) is true for $n = k$. Then
\[
\begin{align*}
x_{k+1} &= \frac{1}{\tan\left(\frac{\pi}{2^{k+1}}\right)} \left( \sqrt{1 + \tan^2\left(\frac{\pi}{2^{k+1}}\right)} - 1 \right) \\
&= \frac{\sec\left(\frac{\pi}{2^{k+1}}\right) - 1}{\tan\left(\frac{\pi}{2^{k+1}}\right)} = \frac{1 - \cos\left(\frac{\pi}{2^{k+1}}\right)}{\sin\left(\frac{\pi}{2^{k+1}}\right)} \\
&= \tan\left(\frac{1}{2} \cdot \frac{\pi}{2^{k+1}}\right) = \tan\left(\frac{\pi}{2^{k+2}}\right),
\end{align*}
\]
so (1) is true for all $n$.

Let $y = 1/2^n$. Then as $n \to \infty$, $y \to 0^+$, so by L’Hôpital’s rule
\[
\lim_{n \to \infty} 2^n x_n = \lim_{y \to 0^+} \frac{\tan(\pi y/2)}{y} = \lim_{y \to 0^+} \frac{\pi}{2} \cdot \frac{\sec^2(\pi y/2)}{1} = \frac{\pi}{2}.
\]

Also solved by H. L. ABBOTT, University of Alberta; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; DUANE M. BROLINE, Eastern Illinois University, Charleston; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinenymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARK KISIN, student, Monash University, Clayton, Australia; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Breda, The Netherlands; KENNETH S. WILLIAMS, Carleton University; and the proposer. Two incorrect solutions were sent in.

All solutions were somewhat like the above. Abbott and the proposer note that $x_n$ is half the side of a regular polygon of $2^{n+1}$ sides which is circumscribed about a unit circle, so $2^n x_n$ will approach $1/4$ of the circumference of this circle. Battles points out the similar problem 1214 of Mathematics Magazine, solution in Vol. 59 (April 1986), pp. 117–118.

Let $A$ and $P$ be points on a circle. Let $l$ be a fixed line through $A$ but not through $P$, and let $x$ be a variable line through $P$ which cuts $l$ at $L_x$ and again at $G_x$. Find the locus of the circumcentre of $\triangle AL_xG_x$.


Let $\theta = \angle AG_xP = \text{arc}(AP)/2$, which is independent of $x$. Let $M_x$ be the circumcentre of $\triangle AG_xL_x$ and $N_x$ the projection of $M_x$ on $l$. Since $\text{arc}(AL_x) = 2\theta$, $\angle AM_xN_x$ is also equal to $\theta$, and so

$$\angle M_xAN_x = \frac{\pi}{2} - \theta,$$

independent of $x$. So all circumcentres lie on one and the same straight line through $A$. [Editor’s note. The diagram illustrates the case that $\theta$ is acute. All this still works, with minor changes, if $G_x$ lies on the minor arc $AP$.]

To construct the locus: when $x$ is parallel to $l$ let $G_x = Q$, so that $PQ$ is parallel to $l$; the circumcircle is then the straight line $AQ$, and the locus the straight line through $A$ perpendicular to $AQ$.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; DAN PEDOE, Minneapolis, Minnesota; D.J. SMEENK, Zaltbommel, The Netherlands; HUME SMITH, Chester, Nova Scotia; and the proposer.

* * * * *


Find a real root of

$$y^5 - 10y^3 + 20y - 12 = 0.$$ 

I. Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

Making the substitution

$$y = 2\sqrt{2}\cosh\theta$$

in the equation of the problem statement and dividing each term by $8\sqrt{2}$ gives

$$16\cosh^5\theta - 20\cosh^3\theta + 5\cosh\theta = \frac{3\sqrt{2}}{4}.$$ 

Using the well-known multiple angle formula for hyperbolic functions, we obtain

$$\cosh 5\theta = \frac{3\sqrt{2}}{4}.$$ 

Now, since

$$\cosh^{-1} u = \ln(u \pm \sqrt{u^2 - 1}),$$
we find

\[ 5\theta = \ln \left( \frac{3\sqrt{2}}{4} \pm \frac{\sqrt{2}}{4} \right) = \pm \frac{1}{2} \ln 2, \]

so \( \theta = \pm 0.1 \ln 2 \). Thus

\[ y = 2\sqrt{2} \cdot \frac{1}{2}(e^{\theta} + e^{-\theta}) = \sqrt{2}(e^{0.1 \ln 2} + e^{-0.1 \ln 2}) = \sqrt{2}(2^{0.1} + 2^{-0.1}) = 2^{0.6} + 2^{0.4}. \]

II. Solution by Murray S. Klaman, University of Alberta.

More generally ([1], [2]), it is known that the equation

\[ y^5 - 5py^3 + 5p^2y - a = 0 \]

is solvable in radicals. Letting \( y = t + p/t \), the equation reduces to

\[ t^5 + \frac{p^5}{t^5} = a \]

so that

\[ 2t^5 = a \pm \sqrt{a^2 - 4p^5}. \]

Here \( p = 2 \) and \( a = 12 \), so \( t^5 = 8 \) or 4 and the real root is

\[ y = 2^{2/5} + 2^{3/5}. \]

Similarly, by using the same substitution, we can solve

\[ y^7 - 7py^5 + 14p^2y^3 - 7p^3y - a = 0 \]

in terms of the seventh roots of unity (due to L. E. Dickson).

References:


Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MATTHEW ENGLANDER, Toronto, Ontario; C. FESTRAETS-HAMOIR, Brussels, Belgium; JEFF HIGHAM, student, University of Toronto; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer. Four other readers sent in approximate solutions (one taken to over 70 decimal places!). There was also one incorrect solution received.


\[ y = x^5 - 10x^3 + 20x - 12 \]
of the problem has the interesting property that its two local maxima have equal $y$-values and its two local minima also have equal $y$-values. According to expert colleague Len Bos, this is true because the above polynomial is actually

$$4\sqrt{8} \cdot T_5(x/\sqrt{8}) - 12,$$

where $T_5$ is the fifth Chebyshev polynomial, which is well known to have this property.


Let $a \leq b < c$ be the lengths of the sides of a right triangle. Find the largest constant $K$ such that

$$a^2(b + c) + b^2(c + a) + c^2(a + b) \geq Kabc$$

holds for all right triangles and determine when equality holds. It is known that the inequality holds when $K = 6$ (problem 351 of the College Math. Journal; solution on p. 259 of Volume 20, 1989).


Let $\theta$ be the angle opposite the side with length $a$; then $0 < \theta \leq \pi/4$. Now

$$\frac{a^2(b + c) + b^2(c + a) + c^2(a + b)}{abc} = \frac{a}{c} + \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} = \sin \theta + \tan \theta + \cot \theta + \cos \theta + \sec \theta + \csc \theta = f(\theta), \text{ say.}$$

Then

$$f'(\theta) = \cos \theta + \sec^2 \theta - \csc^2 \theta - \sin \theta + \sec \theta \tan \theta - \csc \theta \cot \theta,$$

so putting $s = \sin \theta$, $c = \cos \theta$, $f'(\theta) = 0$ implies

$$0 = s^2c^3 + s^2 - c^2 - s^3c^2 + s^3 - c^3 = (s - c)(s + c + s^2 + sc + c^2 - s^2c^2) = (s - c)(s + c + s^2 + sc + c^2).$$

Since each component of the second term is $\geq 0$ for $0 < \theta \leq \pi/4$, and they cannot all be 0 at once, the only turning point of $f(\theta)$ in this range is where $s - c = 0$, i.e., $\theta = \pi/4$. $f(\theta)$ is continuous over $(0, \pi/4]$, and $f(\theta) \to +\infty$ as $\theta \to 0$ from above, therefore $\theta = \pi/4$ gives the lowest possible value for $f(\theta)$ in this range. So

$$K_{\text{max}} = f(\pi/4) = 2 + 3\sqrt{2},$$

and equality holds for the isosceles right-angled triangle.

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; C. FESTRAETS-HAMOIR,
Professor Chalkdust teaches two sections of a mathematics course, with the same material taught in both sections. Section 1 runs on Mondays, Wednesdays, and Fridays for 1 hour each day, and Section 2 runs on Tuesdays and Thursdays for 1.5 hours each day. Normally Professor Chalkdust covers one unit of material per hour, but if she is teaching some material for the second time she teaches twice as fast. The course began on a Monday. In the long run (i.e. after \( N \) weeks as \( N \to \infty \)) will one section be taught more material than the other? If so, which one, and how much more?

Solution by Jeff Higham, student, University of Toronto.

Let \( x \) be the number of units of material covered by Section 1 (from the start of the course) minus the number of units of material covered by Section 2. We will show by induction that, in week \( n \),

\[
x = \begin{cases} 
    \frac{(7 - 2^{5-4n})}{5} & \text{after Monday,} \\
    \frac{(-4 - 2^{1-4n})}{5} & \text{after Tuesday,} \\
    \frac{(3 - 2^{3-4n})}{5} & \text{after Wednesday,} \\
    \frac{(-6 - 2^{2-4n})}{5} & \text{after Thursday,} \\
    \frac{(2 - 2^{1-4n})}{5} & \text{after Friday.}
\end{cases}
\]  

(1) is easily verified when \( n = 1 \). Suppose (1) is true for \( n = k \). Then on Monday of week \( k+1 \), Section 1 covers only “new material” (material not yet covered by the other section), since \( x = \frac{(2 - 2^{1-4k})}{5} > 0 \) on the previous Friday by the induction hypothesis. Thus, after this Monday,

\[
    x = \frac{2 - 2^{1-4k}}{5} + 1 = \frac{7 - 2^{1-4k}}{5} = \frac{7 - 2^{5-4(k+1)}}{5}.
\]

On Tuesday of week \( k+1 \), Section 2 covers the \( \frac{(7 - 2^{5-4(k+1)})}{5} \) units of “old material” (material already covered by the other section) in \( (7 - 2^{5-4(k+1)})/10 < 1.5 \) hours. Thus, for the remaining

\[
    \frac{3}{2} - \frac{7 - 2^{5-4(k+1)}}{10} = \frac{8 + 2^{2-4(k+1)}}{10} = \frac{4 + 2^{4-4(k+1)}}{10}
\]

hours, \( \frac{4 + 2^{4-4(k+1)}}{5} \) units of new material is covered, so \( x = \frac{(-4 - 2^{1-4(k+1)})}{5} \) after this Tuesday. On Wednesday of week \( k+1 \), old material is covered in \( \frac{(4 + 2^{4-4(k+1)})}{10} < 1 \) hours and new material in

\[
    1 - \frac{4 + 2^{4-4(k+1)}}{10} = \frac{6 - 2^{4-4(k+1)}}{10} = \frac{3 - 2^{3-4(k+1)}}{5}
\]
hours, so \(x = (3 - 2^{3-4(k+1)})/5\) after this Wednesday. On Thursday, old material is covered in \((3 - 2^{3-4(k+1)})/10 < 1.5\) hours, and new material in

\[
\frac{3}{2} - \frac{3 - 2^{3-4(k+1)}}{10} = \frac{12 + 2^{3-4(k+1)}}{10} = \frac{6 + 2^{2-4(k+1)}}{5}
\]

hours, so \(x = (-6 - 2^{2-4(k+1)})/5\) after this Thursday. Finally, on Friday, old material is covered in \((6 + 2^{2-4(k+1)})/10 < 1\) hours, and new material in

\[
1 - \frac{6 + 2^{2-4(k+1)}}{10} = \frac{4 - 2^{2-4(k+1)}}{10} = \frac{2 - 2^{1-4(k+1)}}{5}
\]

hours, so \(x = (2 - 2^{1-4(k+1)})/5\) after this Friday. This completes the induction argument.

Clearly as \(n \to \infty\), \(x \to 2/5\) after Friday; therefore in the long run Section 1 is ahead of Section 2 by 2/3 units after Friday.

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; CURTIS COOPER, Central Missouri State University, Warrensburg; RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; HUME SMITH, student, University of British Columbia; and the proposer. One incorrect solution was received which was apparently due to the writer not understanding the problem.

\[\begin{array}{cccccc}
\ast & \ast & \ast & \ast & \ast & \ast
\end{array}\]


\(P\) is any point inside a triangle \(ABC\). Lines \(PA, PB, PC\) are drawn and angles \(PAC, PBA, PCB\) are denoted by \(\alpha, \beta, \gamma\) respectively. Prove or disprove that

\[
cot \alpha + \cot \beta + \cot \gamma \geq \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2},
\]

with equality when \(P\) is the incenter of \(\triangle ABC\).

Solution by Marcin E. Kuczma, Warszawa, Poland.

I choose to disprove.

Lines \(AP, BP, CP\) cut the boundary of triangle \(ABC\) in \(K, L, M\). Let \(U, V\) be the feet of perpendiculars from \(K\) to lines \(AB\) and \(CA\), respectively. Then

\[
cot \alpha - \cot A = \frac{\sin(A - \alpha)}{\sin A \sin \alpha} = \frac{1}{\sin A} \cdot \frac{\sin \alpha}{KU} = \frac{1}{\sin A} \cdot \frac{BK \sin B}{K \sin C}.
\]

Hence (in cyclic notation)
\[ \sum (\cot \alpha - \cot A) = \sum \frac{BK}{KC} \cdot \frac{\sin B}{\sin C \sin A} \geq 3 \left( \prod \frac{BK}{KC} \cdot \frac{\sin B}{\sin C \sin A} \right)^{1/3} = 3(\sin A \sin B \sin C)^{-1/3}, \]  

by the means inequality and Ceva’s theorem. Equality holds if and only if each one of the summands in (1) equals \((\sin A \sin B \sin C)^{-1/3}\), i.e. (writing \(a, b, c\) for the side lengths), when

\[ \frac{BK}{KC} = \left( \frac{ca}{b^2} \right)^{2/3}, \quad \frac{CL}{LA} = \left( \frac{ab}{c^2} \right)^{2/3}, \quad \frac{AM}{MB} = \left( \frac{bc}{a^2} \right)^{2/3}. \]  

In every triangle \(ABC\) there exists a unique point \(P\) for which (2) holds (pick \(K, L, M\) to partition \(BC, CA, AB\) in ratios as in (2); lines \(AK, BL, CM\) are concurrent by the inverse Ceva theorem). For this point the sum \(\sum \cot \alpha\) attains minimum, equal by (1) to

\[ \sum \cot A + 3(\sin A \sin B \sin C)^{-1/3}. \]

When \(P\) is the incenter, the ratios \(BK/KC\), etc. are \(c/b, a/c, b/a\), hence they differ from those of (2), unless the triangle is regular.

For instance, in the isosceles right triangle of vertices \(A = (1, 0), B = (0, 1), C = (0, 0)\), the optimal \(P = (x, y)\) has \(x = 2^{1/3} - 1 = 0.259\ldots, y = 2^{2/3} - 2^{1/3} = 0.327\ldots\) and gives the minimum value

\[ \cot \alpha = 2 + 3 \cdot 2^{1/3} = 5.779\ldots; \]

compare with the incenter \(I = (u, v), u = v = 1 - 2^{-1/2} = 0.292\ldots\), producing

\[ \cot(A/2) = 3 + 2^{1/2} = 5.828\ldots. \]

Also solved by G. P. HENDERSON, Campbellcroft, Ontario.

* * * * *


Evaluate

\[ \lim_{n \to \infty} \prod_{k=3}^{n} \left( 1 - \tan^4 \left( \frac{\pi}{2^k} \right) \right). \]

Solution by Beatriz Margolis, Paris, France.

Observe that

\[ 1 - \tan^4 \left( \frac{\pi}{2^k} \right) = (1 - \tan^2 \left( \frac{\pi}{2^k} \right)) \left( 1 + \tan^2 \left( \frac{\pi}{2^k} \right) \right) \]

\[ = \frac{\cos^2 \left( \frac{\pi}{2^k} \right) - \sin^2 \left( \frac{\pi}{2^k} \right)}{\cos^2 \left( \frac{\pi}{2^k} \right)} \cdot \frac{1}{\cos^2 \left( \frac{\pi}{2^k} \right)} \]

\[ = \frac{\cos \left( \frac{\pi}{2^{k-1}} \right)}{\cos \left( \frac{\pi}{2^k} \right)} \cdot \left( \frac{2 \sin \left( \frac{\pi}{2^k} \right)}{2 \sin \left( \frac{\pi}{2^k} \right) \cos \left( \frac{\pi}{2^k} \right)} \right)^3 \]

\[ = \frac{\cos \left( \frac{\pi}{2^{k-1}} \right)}{\sin \left( \frac{\pi}{2^k} \right)} \cdot \left( \frac{2 \sin \left( \frac{\pi}{2^k} \right)}{\sin \left( \frac{\pi}{2^{k-1}} \right)} \right)^3. \]
Hence
\[
\prod_{k=3}^{n} (1 - \tan^4(\pi/2^k)) = \prod_{k=3}^{n} \cos(\pi/2^{k-1}) \cdot \left( \prod_{k=3}^{n} \frac{2\sin(\pi/2^{k-1})}{\sin(\pi/2^k)} \right)^3
\]
\[
= \frac{\cos(\pi/2^2)}{\cos(\pi/2^n)} \cdot \left( \frac{2^{n-2}\sin(\pi/2^n)}{\sin(\pi/2^2)} \right)^3
\]
\[
= A_n \cdot B_n^3.
\]

Now
\[
\lim_{n \to \infty} A_n = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}
\]
and
\[
\lim_{n \to \infty} B_n = \frac{1}{\sin(\pi/4)} \cdot \frac{\pi}{4} \cdot \lim_{n \to \infty} \frac{\sin(\pi/2^n)}{\pi/2^n} = \frac{\sqrt{2} \pi}{4},
\]
so that
\[
\lim_{n \to \infty} \prod_{k=3}^{n} (1 - \tan^4(\pi/2^k)) = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{2} \pi}{4} \right)^3 = 2 \left( \frac{\pi}{4} \right)^3.
\]

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; CURTIS COOPER, Central Missouri State University, Warrensburg; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALThER JANous, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; T. LEINSTER, Lansing College, England; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Breda, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer. One other reader sent in an approximation.

Janous, Klamkin, Penning, and the proposer in fact prove the more general result
\[
\lim_{n \to \infty} \prod_{k=1}^{n} (1 - \tan^4(\pi/2^k)) = \frac{x^3 \cos x}{\sin^3 x},
\]
which with \(x = \pi/4\) becomes the given problem, and which can be shown as above.

\[\ast \quad \ast \quad \ast \quad \ast \quad \ast\]


Given \(r > 0\), determine a constant \(C = C(r)\) such that
\[
(1 + z)^r (1 + z^r) \leq C(1 + z^2)^r
\]
for all \(z > 0\).
I. Solution by Richard Katz, California State University, Los Angeles.

Let

\[ f(x) = \frac{(1 + x)^r(1 + x^r)}{(1 + x^2)^r} \]

and \( S = S(r) = \sup \{ f(x) \mid x \geq 0 \} \). Clearly any constant \( C \) satisfying \( C \geq S \) will solve the problem and \( S \) is the smallest possible value of \( C \). Now

\[ f(x) = \left( \frac{1 + x}{1 + x^2} \right)^r + \left( \frac{x + x^2}{1 + x^2} \right)^r = (g(x))^r + (h(x))^r, \]

where

\[ g(x) = \frac{1 + x}{1 + x^2}, \quad h(x) = \frac{x + x^2}{1 + x^2} = g\left( \frac{1}{x} \right). \]

Thus \( f(x) = f(1/x) \), and so

\[ S = \sup \{ f(x) \mid 0 \leq x \leq 1 \}. \]

Hence

\[
S = \sup_{0 \leq x \leq 1} \{(g(x))^r + (h(x))^r\}
\leq \sup_{0 \leq x \leq 1} \{(g(x))^r\} + \sup_{0 \leq x \leq 1} \{(h(x))^r\} = \left[ \sup_{0 \leq x \leq 1} g(x) \right]^r + \left[ \sup_{0 \leq x \leq 1} h(x) \right]^r.
\]

Now

\[ \sup\{g(x) \mid 0 \leq x \leq 1\} = g(\sqrt{2} - 1) = \frac{1 + \sqrt{2}}{2} \]

and

\[ \sup\{h(x) \mid 0 \leq x \leq 1\} = h(1) = 1, \]

as are easily checked. Therefore

\[ \left( \frac{1 + \sqrt{2}}{2} \right)^r < S(r) < \left( \frac{1 + \sqrt{2}}{2} \right)^r + 1 \]

and so

\[ C(r) = \left( \frac{1 + \sqrt{2}}{2} \right)^r + 1 \]

works and differs by less than \( 1 \) from the best possible value.

This result can be improved slightly. By writing the equation \( f'(x) = 0 \) in the form

\[ \frac{g(x)}{h(x)} = \left( \frac{h(x)}{g(x)} \right)^{r-1} \]

or

\[ \frac{x^2 + 2r - 1}{x^2 - 2r - 1} = x^{r-1}, \]
the following can be deduced (details are straightforward but tedious):

(i) \( S(r) = 2 \) for \( r \leq 3 \);

(ii) \( S(r) - (1 + \sqrt{2})^r / 2^r \sim 1/2^r \).

Hence for \( r \) sufficiently large (calculations suggest \( r \geq 3 \)), one may take

\[
C(r) = \left( \frac{1 + \sqrt{2}}{2} \right)^r + \frac{1}{2^r r}.
\]

and this differs from the best possible value by less than \( 1/2^r \).

II. Partial solution by Murray S. Klankin, University of Alberta.

Letting \( z = 1 \), it follows that \( C \geq 2 \). We show that for \( C = 2 \), the inequality is at least valid for \( r \leq 3 \).

Letting \( r = 3 \), we have to show that

\[ 2(z^2 + 1)^3 - (z + 1)^3(z^3 + 1) \geq 0. \]

Since the polynomial equals \((z - 1)^3(z^3 - 1) = (z - 1)^4(z^3 + z + 1)\), the inequality is valid. Equivalently,

\[
\frac{1 + z}{2} \cdot \left( \frac{1 + z^3}{2} \right)^{1/3} \leq \frac{1 + z^2}{2}.
\]

We now establish the inequality for \( 0 < r < 3 \), i.e.,

\[
\frac{1 + z}{2} \cdot \left( \frac{1 + z^r}{2} \right)^{1/r} \leq \frac{1 + z^2}{2}.
\]

By the power mean inequality, for \( 0 < r < 3 \),

\[
\frac{1 + z}{2} \cdot \left( \frac{1 + z^r}{2} \right)^{1/r} \leq \frac{1 + z}{2} \cdot \left( \frac{1 + z^3}{2} \right)^{1/3} \leq \frac{1 + z^2}{2}.
\]

The given inequality for \( C = 2 \) and \( r = 4 \) is invalid. For by expanding out we should have

\[ z^8 - 4z^7 + 2z^6 - 4z^5 + 10z^4 - 4z^3 + 2z^2 - 4z + 1 \geq 0. \]

This inequality is invalid since the polynomial factors into

\[(z - 1)^3(z^6 - 2z^5 - 3z^4 - 8z^3 - 3z^2 - 2z + 1),\]

which takes on negative values for \( z \) close to 1.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; DAVID C. VAUGHAN, Wilfrid Laurier University, Waterloo, Ontario; and CHRIS WILDHAGEN, Breda, The Netherlands.
Kuczma and Vaughan show that

\[ C(r) > \left( \frac{1 + \sqrt{2}}{2} \right)^r + \frac{1}{2^r}, \]

(Vaughan only for “large” \( r \), and that “>” could be replaced by “\( \sim \)” as in (ii) of Katz’s proof. Wildhagen predicts the same approximation based on computer results. Kuczma also gives the explicit upper bound

\[ C(r) = \left( \frac{1 + \sqrt{2}}{2} \right)^r \left( 1 + \frac{1}{2^r} \right) \]

for \( r \geq 4 \). All solvers found \( C = 2 \) for \( 0 < r \leq 3 \).

* * * * *

LETTER TO THE EDITOR

Crux readers will be interested to learn that I am in the process of creating an index to the mathematical problems that appear in the problem columns of many mathematical journals. In particular, the problem column from Crux will be indexed in my book. The first volume of my index will cover the years 1980–1984 and will include all problems published in various problem columns in those years, sorted by topic, as well as an author and title index.

As part of the project, I have compiled a list of journals that include problem sections. The list currently contains almost 150 journals from around the world, with bibliographic data. The list includes both current journals and journals that have ceased publication. If any readers would like more information about my indexing project, and/or a free copy of my list of journals with problem columns, they should write to me at the address below.

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* * * * *
Two Theorems on Tangents to a Parabola .......................... Dan Pedoe  97

The Olympiad Corner: No. 124 ................................. R.E. Woodrow  99

Problems: 1631-1640 ......................................................... 113

Solutions: 1513–1522 ......................................................... 115
TWO THEOREMS ON TANGENTS TO A PARABOLA

Dan Pedoe

Let points $P_0, P_1, P_2, \ldots$ on a line $p$ be related by a similarity of the Euclidean plane to points $Q_0, Q_1, Q_2, \ldots$ on a line $q$, that is,

$$P_i P_j : P_j P_k = Q_i Q_j : Q_j Q_k$$

for all $i \neq j \neq k$, and suppose $Q_0 = P_{n_0}$ is the point of intersection of $p$ and $q$, where $n_0 \neq 0$. Then our first theorem was the subject of an earlier article [2] in Crux:

**Theorem I.** The joins $P_i Q_i$ $(i = 0, 1, 2, \ldots)$ envelope a parabola, which also touches $p$ and $q$.

**Proof.** We use some notions of projective geometry (see [1], p. 328): $P_i P_j : P_j P_k$ is the cross-ratio $\{P_i P_k : P_j \infty\}$, so that $\{P_i P_k, P_j \infty\} = \{Q_i Q_k, Q_j \infty\}$, and the range $(P_i)$ is projective with the range $(Q_i)$. Hence since $P_0 \neq Q_0$ ([1], p. 330) the lines $P_i Q_i$ touch a conic. Among these lines is the line $\infty \infty$, which denotes the line at infinity, so that the conic is a parabola. Also $P_0 Q_0$ is the line $p$, and $P_{n_0} Q_{n_0}$ is the line $q$, and $p$ and $q$ also touch the parabola. □

This envelope appears in kindergarten drawings, and is evidently ornamental, but it is sometimes thought, incorrectly, to be the envelope of an hyperbola, or even a circle (see [2]).

Our second theorem turned up more recently in a solution of Crux 1388 [1990: 24], and appears to be distinct from the first, but the two are closely related. It is worth remembering that affine transformations do not change ratios of segments on the same line, or on parallel lines, and that under affine transformations there is essentially only one parabola.

**Theorem II.** A given triangle $T_1 T_2 T_3$ has sides $t_1, t_2, t_3$, and is intersected by a line $t$ in the points $Q_1, Q_2, Q_3$ in order, where $Q_2 Q_3 = k Q_1 Q_2$ for fixed $k$. Then the lines $t$ envelope a parabola, which touches $t_1, t_2$ and $t_3$.

**Proof.** Let $t_4$ be one position of the line $t$. Since a conic is uniquely determined if we are given five tangents, and the line at infinity touches a parabola, there is a unique parabola $P$ which touches $t_1, t_2, t_3$ and $t_4$. Let $P_1, P_2$ and $P_3$ be the respective points of contact of $t_1, t_2$ and $t_3$ with the parabola $P$. Then if $t$ is any tangent to $P$, and the respective intersections of $t$ with $t_1, t_2$ and $t_3$ are $R_1, R_2$ and $R_3$, by [1], p. 330, Theorem II,

$$\{Q_1 Q_3, Q_2 \infty\} = \{P_1 P_3, P_2 \infty\} = \{R_1 R_3, R_2 \infty\},$$

where the first and third cross-ratios are taken on their respective lines, but the second cross-ratio is taken on the parabola $P$: that is, it is the cross-ratio of the pencil.
\(V(P_1P_3, P_2\infty)\), where \(V\) is any point on the parabola \(P\). Hence \(Q_1Q_2 : Q_2Q_3 = R_1R_2 : R_2R_3\), and all tangents \(t\) to the parabola \(P\) intersect the sides of \(T_1T_2T_3\) in points \(R_1, R_2, R_3\) with \(R_2R_3 = kR_1R_2\).

There are no other lines which do this, since a simple *reductio ad absurdum* shows that through a given point \(R_1\) of \(t_1\) there is only one line which intersects \(t_2\) in \(R_2\) and \(t_3\) in \(R_3\) with \(R_2R_3 = kR_1R_2\), and of course there is only one tangent to \(P\) from a point on \(t_1\), this line being itself a tangent. \(\square\)

The connection with Theorem I is clear if we consider a tangent \(t_5\) to \(P\) distinct from \(t_4\) and the sides of triangle \(T_1T_2T_3\), and observe that tangents to the parabola intersect \(t_4\) and \(t_5\) in similar ranges. If \(t'_1, t'_2, t'_3\) are three such tangents, and \(Q'_1, Q'_2, Q'_3\) are the intersections with \(t_4\), and \(R'_1, R'_2, R'_3\) the intersections with \(t_5\), then \(Q'_2Q'_3 = k'Q'_1Q'_2\) and \(R'_2R'_3 = k'R'_1R'_2\), where the value of \(k'\) depends on the points of contact of \(t'_1, t'_2, t'_3\) with the parabola \(P\).

Both theorems, of course, are embraced by the theorem that two given tangents to a parabola are cut in projective ranges by the other tangents to the parabola, and the fact that the points at infinity on each of the given tangents correspond. If we select three points on one given tangent, and the corresponding three points on the other tangent, we are back in the situations considered above.

As a final remark, suppose that we wished to illustrate the case \(k = 1\): that is, given a parabola \(P\), find a triangle \(T_1T_2T_3\) of tangents such that all other tangents intersect the sides of \(T_1T_2T_3\) in points \(Q_1, Q_2\) and \(Q_3\) with \(Q_1Q_2 = Q_2Q_3\). (This was the case used in the proof of *Crux* 1388 [1990: 24].) If the points of contact of the sides of triangle \(T_1T_2T_3\) with the given parabola are \(P_1, P_2\) and \(P_3\), then \(\{P_1P_3, P_2\infty\} = \equiv 1\), so that the lines \(P_1P_3\) and \(P_2\infty\) are conjugate. Let \(P_1\) and \(P_3\) be two points of \(P\) and define \(T_2\) to be the point of intersection of the tangents to \(P\) at \(P_1\) and \(P_3\); draw the line through \(T_2\) and the point at infinity of the parabola (i.e., parallel to the axis of the parabola). If this intersects the parabola at the point \(P_2\), then the tangent to the parabola at \(P_2\) is the side \(T_1T_3\) we are looking for.

References:

Minneapolis, Minnesota
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The first problem set we give is the 9th annual American Invitational Mathematics Examination (A.I.M.E.) written Tuesday, March 19, 1991. The time allowed was 3 hours. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Only the numerical solutions will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588-0322.

1991 AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

1. Find \( x^2 + y^2 \) if \( x \) and \( y \) are positive integers such that

\[
xy + x + y = 71 \quad \text{and} \quad x^2y + xy^2 = 880.
\]

2. Rectangle \( ABCD \) has sides \( AB \) of length 4 and \( CB \) of length 3. Divide \( AB \) into 168 congruent segments with points \( A = P_0, P_1, \ldots, P_{168} = B \), and divide \( CB \) into 168 congruent segments with points \( C = Q_0, Q_1, \ldots, Q_{168} = B \). For \( 1 \leq k \leq 167 \), draw the segments \( P_kQ_k \). Repeat this construction on the sides \( AD \) and \( CD \), and then draw the diagonal \( AC \). Find the sum of the lengths of the 335 parallel segments drawn.

3. Expanding \((1 + 0.2)^{1000}\) by the binomial theorem and doing no further manipulation gives

\[
\binom{1000}{0}(0.2)^0 + \binom{1000}{1}(0.2)^1 + \binom{1000}{2}(0.2)^2 + \cdots + \binom{1000}{1000}(0.2)^{1000} = A_0 + A_1 + \cdots + A_{1000},
\]

where \( A_k = \binom{1000}{k}(0.2)^k \) for \( k = 0, 1, 2, \ldots, 1000 \). For which \( k \) is \( A_k \) the largest?

4. How many real numbers \( x \) satisfy the equation \( \frac{1}{5}\log_2 x = \sin(5\pi x) \)?

5. Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will 20! be the resulting product?

6. Suppose \( r \) is a real number for which

\[
\left| r + \frac{19}{100} \right| + \left| r + \frac{20}{100} \right| + \left| r + \frac{21}{100} \right| + \cdots + \left| r + \frac{91}{100} \right| = 546.
\]
Find $[100r]$. (For real $x$, $[x]$ is the greatest integer less than or equal to $x$.)

7. Find $A^2$, where $A$ is the sum of the absolute values of all roots of the following equation:

$$x = \sqrt{19 + \frac{91}{\sqrt{19 + \frac{91}{\sqrt{19 + \frac{91}{\sqrt{19 + \frac{91}{\sqrt{19 + 1}}}}}}}}}$$

8. For how many real numbers $a$ does the quadratic equation $x^2 + ax + 6a = 0$ have only integer roots for $x$?

9. Suppose that $\sec x + \tan x = \frac{22}{7}$ and that $\csc x + \cot x = \frac{m}{n}$, where $\frac{m}{n}$ is in lowest terms. Find $m + n$.

10. Two three-letter strings, $aaa$ and $bbb$, are transmitted electronically. Each string is sent letter by letter. Due to faulty equipment, each of the six letters has a $1/3$ chance of being received incorrectly, as an $a$ when it should have been a $b$, or as a $b$ when it should have been an $a$. However, whether a given letter is received correctly or incorrectly is independent of the reception of any other letter. Let $S_a$ be the three-letter string received when $aaa$ is transmitted and let $S_b$ be the three-letter string received when $bbb$ is transmitted. Let $p$ be the probability that $S_a$ comes before $S_b$ in alphabetical order. When $p$ is written as a fraction in lowest terms, what is its numerator?

11. Twelve congruent disks are placed on a circle $C$ of radius 1 in such a way that the twelve disks cover $C$, no two of the disks overlap, and so that each of the twelve disks is tangent to its two neighbors. The resulting arrangement of disks is shown in the figure to the right. The sum of the areas of the twelve disks can be written in the form $\pi(a \Leftrightarrow b\sqrt{c})$, where $a$, $b$, $c$ are positive integers and $c$ is not divisible by the square of any prime. Find $a + b + c$.

12. Rhombus $PQRS$ is inscribed in rectangle $ABCD$ so that vertices $P, Q, R,$ and $S$ are interior points on sides $\overline{AB}, \overline{BC}, \overline{CD},$ and $\overline{DA}$, respectively. It is given that $PB = 15$, $BQ = 20$, $PR = 30$, and $QS = 40$. Let $m/n$, in lowest terms, denote the perimeter of $ABCD$. Find $m + n$.

13. A drawer contains a mixture of red socks and blue socks, at most 1991 in all. It so happens that, when two socks are selected randomly without replacement, there is a probability of exactly $1/2$ that both are red or both are blue. What is the largest possible number of red socks in the drawer that is consistent with this data?

14. A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from $A$. 

\[\begin{array}{c}
\text{Diagram of 12 congruent disks arranged on a circle. Each disk is tangent to its neighbors.}
\end{array}\]
15. For positive integer \( n \), define \( S_n \) to be the minimum value of the sum
\[
\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},
\]
where \( a_1, a_2, \ldots, a_n \) are positive real numbers whose sum is 17. There is a unique positive integer \( n \) for which \( S_n \) is also an integer. Find this \( n \).

* 

This month we also give the 1990 Australian Olympiad. I particularly want to thank Andy Liu, University of Alberta, for having collected these, and other problem sets we shall use, while he was at the IMO last summer.

**1990 AUSTRALIAN MATHEMATICAL OLYMPIAD**

Paper I: Tuesday, 13th February, 1990

Time allowed: 4 hours

1. Let \( f \) be a function defined for all real numbers and taking real numbers as its values. Suppose that, for all real numbers \( x, y \), the function \( f \) satisfies
\[
\begin{align*}
(1) \quad f(2x) &= f \left( \sin \left( \frac{\pi x}{2} + \frac{\pi y}{2} \right) \right) + f \left( \sin \left( \frac{\pi x}{2} - \frac{\pi y}{2} \right) \right), \\
(2) \quad f(x^2 - y^2) &= (x + y)f(x - y) + (x - y)f(x + y).
\end{align*}
\]
Show that these conditions uniquely determine
\[
f(1990 + 1990^{1/2} + 1990^{1/3})
\]
and give its value.

2. Prove that there are infinitely many pairs of positive integers \( m \) and \( n \) such that \( n \) is a factor of \( m^2 + 1 \) and \( m \) is a factor of \( n^2 + 1 \).

3. Let \( ABC \) be a triangle and \( k_1 \) be a circle through the points \( A \) and \( C \) such that \( k_1 \) intersects \( AB \) and \( BC \) a second time in the points \( K \) and \( N \) respectively, \( K \) and \( N \) being different. Let \( O \) be the centre of \( k_1 \). Let \( k_2 \) be the circumcircle of the triangle \( KBN \), and let the circumcircle of the triangle \( ABC \) intersect \( k_2 \) also in \( M \), a point different from \( B \). Prove that \( OM \) and \( MB \) are perpendicular.

4. A solitaire game is played with an even number of discs, each coloured red on one side and green on the other side. Each disc is also numbered, and there are two of each number; i.e. \( \{1, 1, 2, 2, 3, \ldots, N, N\} \) are the labels. The discs are laid out in rows with each row having at least three discs. A move in this game consists of flipping over simultaneously two discs with the same label. Prove that for every initial deal or layout there is a sequence of moves that ends with a position in which no row has only red or only green sides showing.
5. In a given plane, let $K$ and $k$ be circles with radii $R$ and $r$, respectively, and suppose that $K$ and $k$ intersect in precisely two points $S$ and $T$. Let the tangent to $k$ through $S$ intersect $K$ also in $B$, and suppose that $B$ lies on the common tangent to $k$ and $K$. Prove: if $\phi$ is the (interior) angle between the tangents of $K$ and $k$ at $S$, then

$$\frac{r}{R} = (2 \sin \frac{\phi}{2})^2.$$

6. Up until now the National Library of the small city state of Sepharia has had $n$ shelves, each shelf carrying at least one book. The library recently bought $k$ new shelves, $k$ being positive. The books will be rearranged, and the librarian has announced that each of the now $n + k$ shelves will contain at least one book. Call a book privileged if the shelf on which it will stand in the new arrangement is to carry fewer books than the shelf which has carried it so far. Prove: there are at least $k + 1$ privileged books in the National Library of Sepharia.

7. For each positive integer $n$, let $d(n)$ be the number of distinct positive integers that divide $n$. Determine all positive integers for which $d(n) = n/3$ holds.

8. Let $n$ be a positive integer. Prove that

$$\frac{1}{\binom{2n}{1}} + \frac{1}{\binom{2n}{2}} + \frac{1}{\binom{2n}{3}} + \cdots + \frac{1}{\binom{2n}{k}} + \cdots + \frac{1}{\binom{2n}{2n-1}} = \frac{1}{n+1}.$$  

Before turning to solutions submitted by the readers, I want to give two comments received about recent numbers of the Corner.


Prove that

$$\frac{x_1^2}{x_1^2 + x_2x_3} + \frac{x_2^2}{x_2^2 + x_3x_4} + \cdots + \frac{x_n^2}{x_n^2 + x_1x_2} \leq n \iff 1$$

where all $x_i > 0$.

Comment by Murray S. Klamkin, University of Alberta.

A more general version of this problem, with solution, appears as Crux 1429 [1990: 155].


Find all pairs of natural numbers $(n, k)$ for which $(n + 1)^k \iff 1 = n!$. 
Editor’s note. In the October 1990 issue of Crux I discussed a solution to this problem, saying it had not been considered in the Corner. Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, takes me to task for not remembering what I have published in earlier numbers! He points out that this problem was also given as number 1 of the 1984 Brazilian Mathematical Olympiad [1987: 70] and that in [1988: 231] I had given George Evagelopoulos’s observation that it was also problem 4 of the 1983 Australian Olympiad, and its solution is in [1986: 23]. So the same problem was used at least three times in three different Mathematical Olympiads, and in three consecutive years! Wang asks if this is a record.

* * * * *

We next give some solutions to “archive problems”. I hope they haven’t been discussed before!

Let $P$ and $Q$ be two polynomials (neither identically zero) with complex coefficients. Show that $P$ and $Q$ have the same roots (with the same multiplicities) if and only if the function $f : \mathbb{C} \to \mathbb{R}$ defined by $f(z) = |P(z)| = |Q(z)|$ has a constant sign for all $z \in \mathbb{C}$ if it is not identically zero.

Solution by Murray S. Klamkin, University of Alberta.
The only if part is easy.
For the if part, we can assume without loss of generality that $f(z) \geq 0$. If $r$ is any root of $P$, it immediately follows that it must also be a root of $Q$ (note if $P$ is a constant, then so also $Q$ is a constant). Also the multiplicity of any root of $P$ must be at most the corresponding multiplicity in $Q$. For if a root $r$ had greater multiplicity in $P$ than in $Q$, by setting $z = r + \epsilon$, where $\epsilon$ is arbitrarily small, we would have $f(z) < 0$. Next the degree of $P$ must be at least the degree of $Q$. Otherwise by taking $|z|$ arbitrarily large, we would have $f(z) < 0$. It follows that $P$ and $Q$ have the same roots with the same multiplicities.

* * * * *

Find all solutions $(x, y, z)$ of the Diophantine equation

$$x^3 + y^3 + z^3 + 6xyz = 0.$$

Comment by Murray S. Klamkin, University of Alberta.
The only nonzero solutions are $(x, y, z) = (1, 1, 0)$ in some order.
Mordell [1] has shown that the equation $x^3 + y^3 + z^3 + dxyz = 0$, for $d \neq 1$, is at most three relatively prime solutions or an infinite number of solutions. Three of the solutions are $(1, 1, 0), (0, 1, 1)$ and $(1, 0, 1)$. (Note that, e.g., $(1, 1, 0)$ and $(1, 1, 0)$ are considered the same.) According to Dickson [2], Sylvester stated that $F \equiv x^3 + y^3 + z^3 + 6xyz$ is not solvable in integers (presumably non-trivially). The same holds for $2F = 27nxyz$ when $27n^2 \equiv 8n + 4$ is a prime, and for $4F = 27nxyz$ when $27n^2 \equiv 36n + 16$ is a prime.
References:


In $\mathbb{R}^n$ let $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, and, for $p \in (0, 1)$, define

$$F_p(X, Y) = \left( \left| \frac{x_1}{p} \right|^{1-p} \left| \frac{y_1}{1 \leftrightarrow p} \right|^{1-p}, \left| \frac{x_2}{p} \right|^{1-p} \left| \frac{y_2}{1 \leftrightarrow p} \right|^{1-p}, \ldots, \left| \frac{x_n}{p} \right|^{1-p} \left| \frac{y_n}{1 \leftrightarrow p} \right|^{1-p} \right).$$

Prove that

$$\|X\|_m + \|Y\|_m \geq \|F_p(X, Y)\|_m,$$

where

$$\|X\|_m = (|x_1|^m + |x_2|^m + \cdots + |x_n|^m)^{1/m}.$$ 

*Solution by Murray S. Klamkin, University of Alberta.*

By Hölder’s inequality

$$\|F_p(X, Y)\| \leq \frac{\|X\|_m^p \|Y\|_m^{1-p}}{p^p(1 \leftrightarrow p)^{1-p}}.$$ 

By the weighted A.M.-G.M. inequality,

$$\|X\|_m + \|Y\|_m = p \left( \frac{\|X\|_m}{p} + (1 \leftrightarrow p) \left( \frac{\|Y\|_m}{p} \right) \right) \geq \frac{\|X\|_m^p \|Y\|_m^{1-p}}{p^p(1 \leftrightarrow p)^{1-p}}.$$ 

The result follows.

If $m \geq 1$, and $x_i, y_i > 0$ for all $i$, then another proof, via Minkowski’s inequality and the weighted A.M.-G.M. inequality, is

$$\|X\|_m + \|Y\|_m \geq \|X + Y\|_m \geq \|F_p(X, Y)\|_m.$$ 

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Determine all values of the real parameter $p$ for which the system of equations

$$x + y + z = 2$$

$$yz + zx + xy = 1$$

$$xyz = p$$

has a real solution.
Solution by Murray S. Klamkin, University of Alberta.

Since 
\[ x = xyz + x^2z + x^2y = p + x^2(2 \Leftrightarrow x) = p + 2x^2 \Leftrightarrow x^3, \]
the solution \((x, y, z)\) is given by the three roots, in any order, of the cubic equation
\[ t^3 \Leftrightarrow 2t^2 + t \Leftrightarrow p = 0. \]

As is known [1], the condition that the general cubic equation
\[ at^3 + 3bt^2 + 3ct + d = 0 \]
have real roots is that \(\Delta \leq 0\) where
\[ \Delta = a^2d^2 \Leftrightarrow 6abcd + 4ac^3 + 4db^3 \Leftrightarrow 3b^2c^2. \]

For the case here,
\[ \Delta = p^2 \Leftrightarrow 4p/27. \]

Consequently, \(p\) must lie in the closed interval \([0, 4/27]\).

References:

*  


In the triangle \(ABC\) with circumcentre \(O\), \(AB = AC\), \(D\) is the midpoint of \(AB\), and \(E\) is the centroid of triangle \(ACD\). Prove that \(OE\) is perpendicular to \(CD\).

Solution by Jie Lou, student, Halifax West High School.
Join the lines \(DE\), \(DO\), and \(AO\), and let \(F\) be the intersection of \(DE\) and \(AC\), \(G\) the intersection of \(AO\) and \(CD\), and \(H\) the intersection of \(AO\) and \(BC\). Find the point \(I\) on \(BC\) such that \(HI = \frac{1}{3}HC\). Since \(\Delta ABC\) is isosceles and \(O\) is the circumcentre, \(AO\) is the central line of \(BC\). Since \(D\) is the midpoint of \(AB\), \(G\) is the centroid of the triangle, and \(GH = \frac{1}{3}AH\). Thus \(GI \parallel AC\). Therefore \(\angle HGI = \angle HAC = \angle DAO\). Since \(O\) is the circumcentre and \(D\) is the midpoint of \(AB\), \(OD\) is perpendicular to \(AB\). Also, we have \(\angle GHI = 90^\circ\). Then \(\Delta GHI \sim \Delta ADO\). From this we have \(GH/AD = H1/DO\). Now, since \(DE = \frac{2}{3}DF = \frac{2}{3}CH = 2HI\) and \(AG = 2GH\), we have
\[ \frac{AG}{AD} = \frac{DE}{DO}. \]

Obviously, \(DE\) is perpendicular to \(AH\), so that \(\angle ODE = 90^\circ \Leftrightarrow \angle ADE = \angle DAG\). From this, \(\Delta ADG \sim \Delta DOE\). Since the angle between \(AD\) and \(DO\) is \(90^\circ\), the angle between \(DG\) and \(EO\) must be \(90^\circ\), too. Thus \(OE\) is perpendicular to \(CD\).
The real numbers \( x_1, x_2, x_3, \ldots \) are defined by

\[
x_1 = a \neq 1 \quad \text{and} \quad x_{n+1} = x_n^2 + x_n \quad \text{for all} \quad n \geq 1.
\]

\( S_n \) is the sum and \( P_n \) is the product of the first \( n \) terms of the sequence \( y_1, y_2, y_3, \ldots \), where

\[
y_n = \frac{1}{1 + x_n}.
\]

Prove that \( aS_n + P_n = 1 \) for all \( n \).

**Solution by Murray S. Klamkin, University of Alberta.**

It follows easily that

\[
\frac{1}{y_{n+1}} = \frac{1}{y_n^2} \leftrightarrow \frac{1}{y_n + 1},
\]

where \( y_1 = 1/(1 + a) \). Now let

\[
\varphi_n = aS_{n+1} + P_{n+1} \leftrightarrow aS_n \leftrightarrow P_n = P_n\left(y_{n+1} \leftrightarrow 1\right) + ay_{n+1}. \tag{2}
\]

Replacing \( n \) by \( n \leftrightarrow 1 \) and dividing gives

\[
\frac{y_n(y_{n+1} \leftrightarrow 1)}{y_n \leftrightarrow 1} = \frac{\varphi_n \leftrightarrow ay_{n+1}}{\varphi_{n-1} \leftrightarrow ay_n}.
\]

It follows from (1) that

\[
\frac{y_n(y_{n+1} \leftrightarrow 1)}{y_n \leftrightarrow 1} = \frac{y_{n+1}}{y_n}.
\]

Hence

\[
\frac{\varphi_n \leftrightarrow ay_{n+1}}{\varphi_{n-1} \leftrightarrow ay_n} = \frac{y_{n+1}}{y_n}
\]

or

\[
\varphi_ny_n = \varphi_{n-1}y_{n+1}. \tag{3}
\]

An easy calculation shows that \( y_2 = 1/(1 + a + a^2) \), so that

\[
\varphi_1 = P_1(y_2 \leftrightarrow 1) + ay_2 = \frac{1}{1 + a} \left( \frac{1}{1 + a + a^2} \leftrightarrow 1 \right) + \frac{a}{1 + a + a^2}
\]

\[
= \frac{1 + a(1 + a)}{(1 + a)(1 + a + a^2)} \leftrightarrow \frac{1}{1 + a} = 0.
\]

Then since \( y_i \neq 0 \) for all \( i \), from (3) \( \varphi_i = 0 \) for all \( i \). Since \( aS_1 + P_1 = 1 \), by (2) \( aS_n + P_n = 1 \) for all \( n \).

Does there exist a function \( f : \mathbb{R} \to \mathbb{R} \) such that \( \lim_{x \to -\infty} f(x) = \infty \) and

\[
\lim_{x \to -\infty} \frac{f(x)}{\ln(\ln(\ldots(\ln x)))} = 0
\]

holds for all \( n \) (where \( n \) is the number of logarithm functions in the denominator)?

Solution by Murray S. Klamkin, University of Alberta

The answer to the given problem is in the affirmative and it is a special case of a result of du Bois-Reymond [1].

First, if \( \frac{f(x)}{g(x)} \to \infty \) as \( x \to \infty \), we say that the order of \( f \) is greater than the order of \( g \) and we write it as \( f \succ g \). The theorem of du Bois-Reymond is that given a scale of increasing functions \( \varphi_n \) such that

\[
\varphi_1 \succ \varphi_2 \succ \varphi_3 \succ \ldots \succ 1,
\]

then there exists an increasing function \( f \) such that \( \varphi_n \succ f \succ 1 \) for all values of \( n \). Here we choose \( \varphi_1 = \ln x, \varphi_2 = \ln \ln x, \varphi_3 = \ln \ln \ln x \), etc.

More generally we have the following: given a descending sequence \( \{\varphi_n\} : \varphi_1 \succ \varphi_2 \succ \varphi_3 \succ \cdots \succ \varphi_n \succ \cdots \succ \varphi \) and an ascending sequence \( \{\psi_n\} : \psi_1 \prec \psi_2 \prec \psi_3 \prec \cdots \prec \psi_p \prec \cdots \prec \psi \) such that \( \psi_p \prec \varphi_n \) for all \( n \) and \( p \) then there is \( f \) such that \( \psi_p \prec f \prec \varphi_n \) for all \( n \) and \( p \).

References:


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Given six segments \( S_1, S_2, \ldots, S_6 \) congruent to the edges \( AB, AC, AD, CD, DB, BC \), respectively, of a tetrahedron \( ABCD \), show how to construct with straightedge and compass a segment whose length equals that of the bialtitude of the tetrahedron relative to opposite edges \( AB \) and \( CD \) (i.e., the distance between the lines \( AB \) and \( CD \)).

Solution by Murray S. Klamkin, University of Alberta.

We use the known construction for an altitude of a tetrahedron [1] and the known theorem [2] that the volume of a tetrahedron equals one-sixth the product of two opposite edges times the sine of the angle between those edges and times the shortest distance between those edges. The volume also equals one-third the product of an altitude and the area of the corresponding face.

Let \( s_1 = |S_1| \), etc. Let \( \theta \) be the angle between edges \( AB \) and \( CD \) and let \( d \) be the distance between \( AB \) and \( CD \). Also let \( h_D \) be the altitude of the tetrahedron from \( D \) and \( h' \) be the altitude of triangle \( ABC \) from \( C \). Then six times the volume of the tetrahedron equals

\[
s_4 s_1 d \sin \theta = 2 h_D [ABC] = h_D s_1 s_2 \sin \angle C AB = h_D s_1 h'.
\]

(1)
Here $[ABC]$ denotes the area of triangle $ABC$. To express $\sin \theta$ as a ratio of two constructible segments, we have

$$s_4 s_1 \cos \theta = |(A \leftrightarrow B) \cdot C| = |A \cdot C \leftrightarrow B \cdot C| = |s_3 s_4 \cos \angle ADC \leftrightarrow s_5 s_4 \cos \angle BDC|,$$  \hspace{1cm} (2)

where $A, B, C$ are respective vectors from $D$ to $A, B, C$. Now if $AA'$ and $BB'$ are the respective altitudes in triangles $ACD$ and $BCD$, then $s_3 \cos \angle ADC = A'D$ and $s_5 \cos \angle BDC = B'D$. Hence from (2),

$$\cos \theta = \frac{|A'D \leftrightarrow B'D|}{s_1} = \frac{A'B'}{s_1},$$

where the length $A'B'$ is constructible. From this, we easily can obtain $\sin \theta = \alpha/s_1$ where $\alpha = \sqrt{s_1^2 \sin^2 (A'B')^2}$ is also constructible. Finally from (1),

$$d = \frac{h_D h'/s_1}{s_4 \alpha}$$

which is constructible in the usual way.

References:


We are given a unit circle $C$ with center $M$ and a closed convex region $R$ in the interior of $C$. From every point $P$ of circle $C$, there are two tangents to the boundary of $R$ that are inclined to each other at $60^\circ$. Prove that $R$ is a closed circular disk with center $M$ and radius $1/2$.

Solution by Murray S. Klamkin, University of Alberta.

Let $P$ be on $C$ and let the two tangents from $P$ to $R$ meet $C$ again at $Q$ and $Q'$. Let the other tangent from $Q(Q')$ meet the circle again at $S(S')$ respectively. Since angles $P, Q, Q'$ are all $60^\circ$, $QS$ must coincide with $Q'S'$ so that $PQS$ is an equilateral triangle. Hence $R$ is the envelope of all inscribed equilateral triangles in a circle. This envelope is known to be a circle concentric with $C$ and with radius half that of $C$. 

* * *
We finish this month’s Corner with solutions to four of the five problems of the 1989 Asian Pacific Mathematical Olympiad [1989: 131]. We’re still waiting for a correct proof of number 2.

1. Let \(x_1, x_2, \ldots, x_n\) be positive real numbers, and let \(S = x_1 + x_2 + \cdots + x_n\). Prove that

\[
(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S^2}{2!} + \cdots + \frac{S^n}{n!}.
\]

**Correction and solution by George Evagelopoulos, Athens, Greece, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.**

The correct inequality should be

\[
(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S^2}{2!} + \cdots + \frac{S^n}{n!}.
\]

This is well known and can be found, for example, in Analytic Inequalities by D.S. Mitrinović (§§3.2.42). Now by the AM-GM inequality

\[
(1 + x_1) \cdots (1 + x_n) \leq \left( \frac{n + x_1 + \cdots + x_n}{n} \right)^n = \left( 1 + \frac{S}{n} \right)^n
\]

\[
= 1 + n \left( \frac{S}{n} \right) + \frac{n(n-1)}{2} \left( \frac{S}{n} \right)^2 + \cdots + \left( \frac{S}{n} \right)^n
\]

using the binomial theorem. Since \((n \leftrightarrow m)!n^m \geq n!\), the coefficient of \(S^m\) is

\[
\binom{n}{m} \frac{1}{n^m} = \frac{n!}{m!(n \leftrightarrow m)!n^m} \leq \frac{n!}{m!n!} = \frac{1}{m!}
\]

from which the result is immediate.

3. Let \(A_1, A_2, A_3\) be three points in the plane, and for convenience, let \(A_4 = A_1, A_5 = A_2\). For \(n = 1, 2, 3\) suppose that \(B_n\) is the midpoint of \(A_nA_{n+1}\), and suppose that \(C_n\) is the midpoint of \(A_nB_n\). Suppose that \(A_nC_{n+1}\) and \(B_nA_{n+2}\) meet at \(D_n\), and that \(A_nB_{n+1}\) and \(C_nA_{n+2}\) meet at \(E_n\). Calculate the ratio of the area of triangle \(D_1D_2D_3\) to the area of triangle \(E_1E_2E_3\).

**Solution by George Evagelopoulos, Athens, Greece.**

Let \(X\) denote the centroid of \(\triangle A_1A_2A_3\). Then

\[
B_1X = \frac{1}{3} A_3B_1, \quad B_2X = \frac{1}{3} A_1B_2, \quad B_3X = \frac{1}{3} A_2B_3.
\]

Using perspectives from \(A_1\) the cross-ratios \((A_2, A_3; B_2, C_2)\) and \((B_1, A_3; X, D_1)\) are equal. Therefore

\[
\frac{A_2B_2}{B_2A_3} \cdot \frac{C_2A_3}{A_2C_2} = \frac{B_1X}{XA_3} \cdot \frac{D_1A_3}{B_1D_1}
\]

from which

\[
1 \cdot 3 = \frac{1}{2} \cdot \frac{D_1A_3}{B_1D_1},
\]
This gives $D_1A_3 = 6B_1D_1$ and so $B_1D_1 = \frac{1}{7}B_1A_3$. Hence

$$\frac{D_1X}{B_1X} = \frac{B_1X}{B_1X} \iff \frac{D_1}{B_1} \iff \frac{1}{7} = \frac{4}{7}.$$

Similarly $D_2X/B_2X = \frac{4}{7}$ and $\Delta D_1D_2X$ is similar to $\Delta B_1B_2X$. Thus

$$\frac{[D_1D_2X]}{[B_1B_2X]} = \left(\frac{4}{7}\right)^2 = \frac{16}{49},$$

where $[T]$ denotes the area of triangle $T$. In similar fashion

$$\frac{[D_2D_3X]}{[B_2B_3X]} = \frac{[D_1D_3X]}{[B_1B_3X]} = \frac{16}{49}.$$

It follows that

$$[D_1D_2D_3] = \frac{16}{49}[B_1B_2B_3] = \frac{4}{49}[A_1A_2A_3],$$

using the fact that triangles $B_1B_2B_3, B_3A_1B_1, B_2B_1A_2$ and $A_3B_3B_2$ are congruent.

Now, with $A_2$ as the centre

$$(A_1, A_3; B_3, C_3) = (B_1, A_3; X, E_3)$$

and so

$$1 \cdot \frac{1}{3} = \frac{A_1B_3}{B_3A_3} \cdot \frac{C_3A_3}{A_1C_3} = \frac{B_1X}{XA_3} \cdot \frac{E_3A_3}{B_1E_3} = \frac{1}{2} \cdot \frac{E_3A_3}{B_1E_3}.$$

Since $XA_3 = \frac{2}{3}B_1A_3$, this means

$$XE_3 = \left[\frac{2}{3} \iff \frac{2}{5}\right] \cdot \frac{1}{5}B_1A_3 = \frac{4}{15}B_1A_3.$$

Hence $XE_3 = \frac{2}{5}XA_3$ and $[XE_3E_1] = \frac{1}{25}[XA_3A_1]$. Finally $[E_1E_2E_3] = \frac{1}{25}[A_1A_2A_3]$ so that

$$[D_1D_2D_3] = \frac{25}{49}[E_1E_2E_3].$$

4. Let $S$ be a set consisting of $m$ pairs $(a, b)$ of positive integers with the property that $1 \leq a < b \leq n$. Show that there are at least

$$4m \cdot m \iff n^2 / 4$$

triples $(a, b, c)$ such that $(a, b), (a, c)$ and $(b, c)$ belong to $S$.

Solution by George Evagelopoulos, Athens, Greece.

Draw a graph whose vertices are the integers $1, 2, \ldots, n$ with an edge between $x$ and $y$ if and only if either $(x, y)$ or $(y, x)$ belongs to $S$. Those 3-element subsets of $S$ in which
any two elements are connected by an edge will be called “triangles” and the number of edges to which a vertex \( x \) belongs will be denoted by \( d(x) \).

For example, if \( S = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\} \) where \( n = 4 \), the subsets \( \{(1, 2), (1, 3), (2, 3)\} \) and \( \{(2, 3), (2, 4), (3, 4)\} \) form triangles as illustrated. Also \( d(2) = 3 \), for example. The problem is to determine a lower bound for the number of triangles in the general case.

Let \( x \) be joined to \( y \) by an edge. Then \( d(x) + d(y) \) \( \equiv \) 2 edges are attached to the remaining \( n \equiv 2 \) vertices and therefore at least \( d(x) + d(y) \equiv 2 \equiv (n \equiv 2) \) vertices are attached to both \( x \) and \( y \). Hence at least \( d(x) + d(y) \equiv n \) triangles contain both \( x \) and \( y \). It follows that the total number of triangles is at least

\[
\sum_{(x,y) \in S} \frac{d(x) + d(y) \equiv n}{3},
\]

since each triangle is counted three times in the sum. Now

\[
\sum_{(x,y) \in S} (d(x) + d(y)) = \sum_{x=1}^{n} d(x)^2
\]

because each \( d(x) \) occurs exactly \( d(x) \) times in the sum over the elements of \( S \). Therefore, by Chebyshev’s inequality, we get

\[
\sum_{(x,y) \in S} \frac{d(x) + d(y) \equiv n}{3} = \frac{1}{3} \left( \sum_{x=1}^{n} d(x)^2 \equiv nm \right)
\]

\[
\geq \frac{1}{3} \left( \frac{1}{n} \left( \sum_{x=1}^{n} d(x) \right)^2 \equiv nm \right)
\]

\[
= \frac{4m(m \equiv n^2/4)}{3n}
\]

because \( \sum_{x=1}^{n} d(x) = 2m \).

5. Determine all functions \( f \) from the reals to the reals for which

(i) \( f(x) \) is strictly increasing,
(ii) \( f(x) + g(x) = 2x \) for all real \( x \) where \( g(x) \) is the composition inverse function to \( f(x) \). (Note: \( f \) and \( g \) are said to be composition inverses if \( f(g(x)) = x \) and \( g(f(x)) = x \) for all real \( x \).)

Solution by George Evagelopoulos, Athens, Greece (with an assist by the editors).

We will prove that \( f(x) = x + d \) for some constant \( d \), i.e., \( f(x) \equiv x \) is a constant function.

For each real \( d \), denote by \( S_d \) the set of all \( x \) for which \( f(x) \equiv x = d \). Then we must show that exactly one \( S_d \) is nonempty. First we prove two lemmas.
Lemma 1. If \( x \in S_d \) then \( x + d \in S_d \).

Proof. Suppose \( x \in S_d \). Then \( f(x) = x + d \), so \( g(x+d) = x \), and \( f(x+d) + g(x+d) = 2x + 2d \). Therefore \( f(x + d) = x + 2d \) and \( x + d \in S_d \). \( \square \)

Lemma 2. If \( x \in S_d \) and \( y \geq x \) then \( y \not\in S_d \) for any \( d' < d \).

Proof. First let \( y \) satisfy \( x \leq y < x + (d \leftrightarrow d') \). Note that by monotonicity \( f(y) \geq f(x) = x + d \). Hence \( y \in S_d \) would imply \( y + d' \geq x + d \) or \( y \geq x + (d \leftrightarrow d') \), a contradiction. Thus \( y \not\in S_d \) in this case. Now by induction it follows that for all \( x \in S_d \),

\[
\text{if } x + (k \leftrightarrow 1)(d \leftrightarrow d') \leq y < x + k(d \leftrightarrow d') \text{ then } y \not\in S_d.
\]

The base case \( k = 1 \) is proved above. Assume the statement holds for some \( k \) and let

\[
x + k(d \leftrightarrow d') \leq y < x + (k + 1)(d \leftrightarrow d').
\]

Then

\[
x + d + (k \leftrightarrow 1)(d \leftrightarrow d') \leq y + d' < x + d + k(d \leftrightarrow d').
\]

But \( x + d \in S_d \), and so by the induction hypothesis \( y + d' \not\in S_d \). The lemma follows. \( \square \)

Now suppose that two \( S_d \)'s are nonempty, say \( S_d \) and \( S_{d'} \) where \( d' < d \). If \( 0 < d' \), then \( S_{d'} \) must contain arbitrarily large numbers by Lemma 1. But this is impossible by Lemma 2.

Editor's note. The above, slightly rewritten, is Evagelopoulos's argument, except he hadn't noted that his argument required \( 0 < d' \). We now complete the proof.

Lemma 3. If \( S_d \) and \( S_{d'} \) are nonempty and \( d' < d'' < d \) then \( S_{d''} \) is also nonempty.

Proof. Since \( S_d \) and \( S_{d'} \) are nonempty, there are \( x \) and \( x' \) so that \( f(x) \leftrightarrow x = d \) and \( f(x') \leftrightarrow x' = d' \). Since \( f \) is increasing and has an inverse, it is continuous, so the function \( f(x) \leftrightarrow x \) is also continuous. Thus by the Intermediate Value Theorem there is \( x'' \) so that \( f(x'') \leftrightarrow x'' = d'' \), so \( S_{d''} \neq \emptyset \). \( \square \)

Now by Lemma 3 we need only consider two cases: \( 0 < d' < d \), which was handled by Evagelopoulos, and \( d' < d < 0 \). We do the second case. Choose some \( y \in S_{d'} \). By Lemma 1, \( S_d \) contains arbitrarily large negative numbers, so we can find \( x \in S_d \) such that \( x < y \). But then \( y \not\in S_d \) by Lemma 2. This contradiction completes the proof.

* * * * *

That's all the room we have this issue. Send me your contests and nice solutions!
PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1991, although solutions received after that date will also be considered until the time when a solution is published.

1631*. Proposed by Murray S. Klamkin, University of Alberta. (Dedicated to Jack Garfunkel.)

Let $P$ be a point within or on an equilateral triangle and let $c_1, c_2, c_3$ be the lengths of the three concurrent cevians through $P$. Determine the largest constant $\lambda$ such that $c_1^\lambda, c_2^\lambda, c_3^\lambda$ are the sides of a triangle for any $P$.

1632. Proposed by Stanley Rabinowitz, Westford, Massachusetts.

Find all $x$ and $y$ which are rational multiples of $\pi$ (with $0 < x < y < \pi/2$) such that $\tan x + \tan y = 2$.

1633. Proposed by Toshio Seimiya, Kawasaki, Japan.

In triangle $ABC$, the internal bisectors of $\angle B$ and $\angle C$ meet $AC$ and $AB$ at $D$ and $E$, respectively. We put $\angle BDE = x, \angle CED = y$. Prove that if $\angle A > 60^\circ$ then $\cos 2x + \cos 2y > 1$.


A cafeteria at a university has round tables (of various sizes) and rectangular trays (all the same size). Diners place their trays of food on the table in one of two ways, depending on whether the short or long sides of the trays point toward the centre of the table:

Moreover, at the same table everybody aligns their trays the same way. Suppose $n$ mathematics students come in to eat together. How should they align their trays so that the table needed is as small as possible?
1635. Proposed by Jordi Dou, Barcelona, Spain.
Given points $B_1, C_1, B_2, C_2, B_3, C_3$ in the plane, construct an equilateral triangle $A_1A_2A_3$ so that the triangles $A_1B_1C_1, A_2B_2C_2$ and $A_3B_3C_3$ are directly similar.

1636*. Proposed by Walther Janous, Ursulinegymnasium, Innsbruck, Austria.
Determine the set of all real exponents $r$ such that

$$d_r(x, y) = \frac{|x - y|}{(x + y)^r}$$

satisfies the triangle inequality

$$d_r(x, y) + d_r(y, z) \geq d_r(x, z) \quad \text{for all } x, y, z > 0$$

(and thus induces a metric on $\mathbb{R}^+$—see Crux 1449, esp. [1990: 224]).

1637. Proposed by George Tsintsifas, Thessaloniki, Greece.
Prove that

$$\sum \frac{\sin B + \sin C}{A} > \frac{12}{\pi}$$

where the sum is cyclic over the angles $A, B, C$ (measured in radians) of a nonobtuse triangle.

1638. Proposed by Juan C. Candeal, Universidad de Zaragoza, and Esteban Indurain, Universidad Pública de Navarra, Pamplona, Spain.
Find all continuous functions $f : (0, \infty) \to (0, \infty)$ satisfying the following two conditions:

(i) $f$ is not one-to-one;
(ii) if $f(x) = f(y)$ then $f(tx) = f(ty)$ for every $t > 0$.

$ABCD$ is a convex cyclic quadrilateral. Prove that

$$(AB + CD)^2 + (AD + BC)^2 \geq (AC + BD)^2.$$  

Find

$$\lim_{n \to \infty} \left( \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1} \right).$$

* * * * * * *
SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


(a) A planar centrosymmetric polygon is inscribed in a strictly convex planar centrosymmetric region $R$. Prove that the two centers coincide.

(b) Do part (a) if the polygon is circumscribed about $R$.

(c) Do (a) and (b) still hold if the polygon and region are $n$-dimensional for $n > 2$?

II. Solution to (a) and (b) by Jordi Dou, Barcelona, Spain.

(a) Let $M$ be the centre of the polygon $P$ and $O$ the centre of $R$. Let $AB$ be a side of $P$ and $ST$ the symmetrically opposite side with respect to $M$. Let $A'B'$ be symmetric to $AB$ with respect to $O$. If $O$ and $M$ are distinct, we have three chords $AB, TS$ and $B'A'$ of $R$ which are of equal length and are parallel, which is incompatible with the strict convexity of $R$.

(b) Let $a$ be a side of $P$ and $s$ the opposite side. Let $a'$ be symmetric to $a$ with respect to $O$. If $O$ and $M$ are distinct and $a$ is chosen not parallel to $OM$, we have three tangents to $R$ which are mutually parallel, which is again incompatible with the strict convexity of $R$.

II. Solution to (b) and (c) by Marcin E. Kuczma, Warszawa, Poland.

[Kuczma also solved part (a). — Ed.]

(b) Suppose the polygon is circumscribed about $R$. Choose a pair of parallel sides. They are contained in two parallel supporting lines of $R$. To every direction there exists exactly one pair of supporting lines, situated symmetrically relative to $O$, the center of symmetry of $R$. Thus $O$ lies midway between (the lines containing) the chosen sides. Repeat the argument taking another pair of parallel sides; point $O$ is also equidistant from these sides. The only point with these properties is the center of symmetry of the polygon. Note that strict convexity is not needed in this part.

(c) The analogue of part (b) is true in any dimension $n \geq 2$. The proof is the same as above; just write hyperplanes for lines; and instead of two pairs of opposite sides consider $n$ pairs of opposite faces of the polyhedron.

The analogue of part (a) is false in every dimension $n \geq 3$. Here is a counterexample.

Consider the points (in $\mathbb{R}^n$)

\[ A_i = (q, \ldots, q, \cancel{1}, q, \ldots, q) \quad B_i = (\cancel{1}, q, \ldots, q, \cancel{1}, \cancel{1}, q, \ldots, q) \]

for $i = 1, \ldots, n$, where $q$ is the root of the equation

\[ q^3 + 3q = \frac{4}{n \cancel{1}}; \quad \text{(1)} \]

Note that

\[ \frac{1}{n \cancel{1}} < q < 1 \quad \text{(2)} \]
Let $d$ be the metric in $\mathbb{R}^n$ induced by the 3-norm

$$\|X\|_3 = \left(\sum_{i=1}^{n} |x_i|^3\right)^{1/3}$$

for $X = (x_1, \ldots, x_n)$. [Editor’s note: that is, the “distance” from $X$ to $Y$ is defined to be $\|X - Y\|_3$.] By (1),

$$d(A_i, C)^3 = (n \leftrightarrow 1)(1 \leftrightarrow q)^3 + 2^3 = (n \leftrightarrow 1)(1 + 3q^2) + 4$$

and also

$$d(B_i, C)^3 = (n \leftrightarrow 1)(1 + q)^3 + 0^3 = (n \leftrightarrow 1)(1 + 3q^2) + 4.$$  

Points $A_1, \ldots, A_n, B_1, \ldots, B_n$ span a polyhedron symmetric about [and so centered at] the origin. Since $A_1, \ldots, A_n$ lie on the hyperplane $\sum X_i = (n \leftrightarrow 1)q \leftrightarrow 1$, which by (2) does not pass through the origin, the polyhedron is nondegenerate. In view of (3) and (4) it is inscribed in a certain $d$-ball, centered at $C$, not at the origin; and this $d$-ball is certainly a strictly convex centro symmetric region.

Also solved (part (a)) by WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; and (parts (a) and (b)) the proposer. A further reader sent in an incorrect answer due to misunderstanding the problem.

* * * * * * *


Let $ABC$ be a triangle with sides $a, b, c$ and let $P$ be a point in the same plane. Put $AP = R_1, BP = R_2, CP = R_3$. It is well known that there is a triangle with sides $aR_1, bR_2, cR_3$. Find the locus of $P$ so that the area of this triangle is a given constant.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Inversion with center $P$ and inversion constant $k$ takes points $A, B, C$ to $A', B', C'$ such that

$$\frac{PA'}{PB} = \frac{k}{PA \cdot PB} = \frac{PB'}{PA} \quad \text{(and cyclically).}$$

So the triangles $PAB$ and $PB'A'$ are similar, in the above ratio, which is therefore also equal to the ratio $A'B'C'/AB$. Denoting the sides of triangle $A'B'C'$ by $a', b', c'$ we thus have

$$\frac{a'}{a} = \frac{k}{R_2R_3}, \quad \frac{b'}{b} = \frac{k}{R_3R_1}, \quad \frac{c'}{c} = \frac{k}{R_1R_2}. \quad \text{(1)}$$

So if we take

$$k = R_1R_2R_3, \quad \text{(2)}$$

the resulting triangle $A'B'C'$ will be the one under consideration.

Let $O$ be the circumcenter of triangle $ABC$, let $R$ and $R'$ be the circumradii of triangles $ABC$ and $A'B'C'$, and let $F$ and $F'$ be the areas of these triangles. By the
properties of inversion (see e.g. N. Altshiller Court, *College Geometry*, New York, 1962, §§526 and 413),

$$R' = \frac{k}{|R^2 - d^2|},$$

where \(d = OP\) — provided \(d \neq R\), of course.

From (1)—(3) and from

$$F = \frac{abc}{4R}, \quad F' = \frac{d'd'c'}{4R'},$$

we obtain

$$\frac{F'}{F} = \frac{|R^2 - d^2|}{R}.$$  \hspace{1cm} (4)

Hence \(F'\) shall assume a constant value \(t^2F\) \((t > 0)\) if and only if \(R^2 - d^2 = \pm t^2\). So the conclusion is:

if \(t > R\), the locus of \(P\) is the circle of center \(O\) and radius \(\sqrt{r^2 + t^2}\);

if \(t \leq R\), the locus is the union of two circles centered at \(O\) and of radii \(\sqrt{R^2 \pm t^2}\)
(the smaller degenerating to the point \(O\) when \(t = R\)).

*Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.*

The fact that for some given areas the locus of \(P\) consists of two concentric circles was only pointed out by Kuczma and the proposer. Klamkin's proof was rather like the above but obtained equation (4) by referring to his papers

*Triangle inequalities via transforms, Notices of Amer. Math. Soc., January 1972, A-103,104; and

An identity for simplexes and related inequalities, Simon Stevin 48e (1974-75) 57-64; or alternatively, to p. 294 of Mitrinović et al, Recent Advances in Geometric Inequalities, and various places in Crux, e.g., [1990: 92].

* * * * *


We are given a finite collection of segments in the plane, of total length 1. Prove that there exists a straight line \(l\) such that the sum of lengths of projections of the given segments to line \(l\) is less than \(2/\pi\).

*Solution by Jordi Dou, Barcelona, Spain.*

We translate the \(n\) segments \(s_i\) \((1 \leq i \leq n)\) so that their midpoints all coincide at a point \(V\) (Figure 1). Designate the 2\(n\) endpoints, ordered cyclically, by \(A_1, A_2, \ldots, A_n, A'_1, A'_2, \ldots, A'_n\). Starting at a point \(P''_1\) we draw segment \(P''_1P'_1\) equal and parallel to \(VA_1\), then starting at \(P_1\) we draw segment \(P_1P_2\) equal and parallel to \(VA_2\), and so on to obtain \(P_3, \ldots, P_n, P'_1, \ldots, P'_n\). We obtain (Figure 2) a convex polygon \(P\) of 2\(n\) sides, with a centre \(O\) of symmetry (because the pairs of opposite sides are equal and parallel).
Choose a pair of opposite sides whose distance apart \((D)\) is minimal. Consider the circle \(\omega\) of centre \(O\) and diameter \(D\); it is tangent to the two opposite sides at interior points \(T\) and \(T'\), and thus is interior to \(P\). Then

\[
\pi D < \text{perimeter of } P = \sum_{i=1}^{n} s_i = 1,
\]

so \(D < 1/\pi\). Therefore the orthogonal projection of all \(2n\) sides of \(P\) onto \(TT'\) has total length \(2D < 2/\pi\), as was to be proved.

Note that \(TT'\) is the line for which the sum of projections is minimal. It is also clear that the largest diagonal of \(P\) gives the direction of maximal sum of projections of the \(n\) segments.

Also solved (almost the same way!) by MURRAY S. KLMKIN, University of Alberta; T. LEINSTER, Lancing College, England; P. PENNING, Delft, The Netherlands; and the proposer.

The proposer did give a second proof using integration, and suggests the analogous problem for segments of total length 1 in three-dimensional space. With integration he obtains that there is a line so that the sum of the lengths of the projections of the segments to this line is less than 1/2 (and this is best possible, as was \(2/\pi\) in the two-dimensional case). He would like a simpler proof. Might there also be a generalization to planar regions in three-space?

The problem was suggested by the proposer for the 1989 IMO, but not used (see #73, p. 45 of the book 30th International Mathematical Olympiad, Braunschweig 1989—Problems and Results, which was reviewed in February).

$ABC$ is an isosceles triangle in which $AB = AC$ and $\angle A < 90^\circ$. Let $D$ be any point on segment $BC$. Draw $CE$ parallel to $AD$ meeting $AB$ produced in $E$. Prove that $CE > 2CD$.

I. Solution by Jack Garfunkel, Flushing, N.Y.

Draw $AF$ parallel to $BC$ so that $AF = CD$. Draw $AM = MN$ where $M$ is the midpoint of $CE$. Then $AENC$ is a parallelogram. Since $\angle A < 90^\circ$, $\angle EAC$ is an obtuse angle. Thus, diagonal $CE >$ diagonal $AN = 2AM$. Also $AM \geq AF$, since a median is $\geq$ the angle bisector drawn from the same vertex. So we have proved the stronger inequality $CE \geq 2AM \geq 2AF = 2CD$.

II. Solution by T. Leinster, Lancing College, England.

Triangle $CEB$ is similar to triangle $DAB$, so $CE = BC \cdot DA/BD$. Thus (since $BC = CD + BD$)

\[
\frac{CE}{CD} = \frac{BC \cdot DA}{CD \cdot BD} \geq \frac{BC \cdot DA}{BC^2/4} = \frac{4DA}{BC}
\]

\[
\geq \frac{4}{BC} \cdot \frac{BC}{2} \tan \angle ABC
\]

\[
> 2 \quad \text{(since } \angle ABC > 45^\circ).\]

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, and JOSÉ YUSTY PITA, Madrid, Spain; JORDI DOU, Barcelona, Spain; L.J. HUT, Groningen, The Netherlands; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; and the proposer. Two incorrect solutions were also sent in.

Kuczma’s solution was very similar to solution II. Several solvers pointed out that $CE = 2CD$ holds if $\angle A = 90^\circ$ and $D$ is the midpoint of $BC$.

* * * * *


[This problem came up in a combinatorics course, and is quite likely already known. What is wanted is a nice answer with a nice proof. A reference would also be welcome.]

Imagine you are standing at a point on the edge of a half-plane extending infinitely far north, east, and west (say, on the Canada–USA border near Estevan, Saskatchewan). How many walks of $n$ steps can you make, if each step is 1 metre either north, east, west, or south, and you never step off the half-plane? For example, there are three such walks of length 1 and ten of length 2.
I. Solution by Mike Hirschhorn, University of New South Wales, Kensington, Australia.

First we note: the number of walks of \( n \) (unit) steps on \( \mathbb{Z}^+ \) (the nonnegative integers) starting at 0 and ending at \( n \equiv 2k \) is

\[
\binom{n}{k} \equiv \binom{n}{k} \equiv 1, \quad 0 \leq k \leq \lfloor n/2 \rfloor.
\]

(Editor’s note. This can be proved by the familiar (but worth repeating) “reflection principle”. The total number of \( n \)-step walks on \( \mathbb{Z} \) starting at 0 and ending at \( n \equiv 2k \) equals the number of \( n \)-sequences of \( n \equiv k \) R’s (rights) and \( k \) L’s (lefts), which is \( \binom{n}{k} \). From this we wish to subtract the “bad” walks, i.e., the \( n \)-step walks from 0 to \( n \equiv 2k \) which land on \( \equiv 1 \) at some point. By the reflection principle these correspond to the \( n \)-walks from \( \equiv 2 \) to \( n \equiv 2k \), i.e., the number of \( n \)-sequences of \( n \equiv k + 1 \) R’s and \( k \equiv 1 \) L’s; there are \( \binom{n}{k-1} \) of these, so (1) follows. The correspondence of the above walks is obtained by interchanging R’s and L’s on that part of the walk from 0 to the first time \( \equiv 1 \) is reached. Drawing walks as “graphs” (as shown), this amounts to “reflecting” an initial piece of the graph about the horizontal line \( y = \equiv 1 \), giving the dotted graph. Back to Hirschhorn’s proof!)

A corollary is: the number of walks of \( n \) steps on \( \mathbb{Z}^+ \) starting at 0 is

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( \binom{n}{k} \equiv \binom{n}{k} \equiv 1 \right) = \binom{n}{\lfloor n/2 \rfloor}
\]

(follows by telescoping).

Finally we prove that walks of \( n \) unit steps on \( \mathbb{Z} \times \mathbb{Z}^+ \) (i.e., the integer points of the upper half-plane) starting at \((0, 0)\) are equinumerous with walks of \( 2n + 1 \) steps on \( \mathbb{Z}^+ \) starting at 0.

To see this, consider a walk of \( n \) steps on \( \mathbb{Z} \times \mathbb{Z}^+ \). It can be written as a sequence of moves L, R, U(up), D(down). [At each stage the number of U’s must equal or exceed the number of D’s.] Replace each L by LR, R by RL, U by RR, D by LL, and prefix this sequence with an R. This process gives a walk of length \( 2n + 1 \) on \( \mathbb{Z}^+ \), and the process is reversible.

Therefore the number of walks of \( n \) steps on \( \mathbb{Z} \times \mathbb{Z}^+ \) starting at \((0, 0)\) is, by (2),

\[
\binom{2n + 1}{n}.
\]

(Editor’s note. Hirschhorn sent in a second proof, based on a formula for the number of walks of \( n \) steps on \( \mathbb{Z} \times \mathbb{Z}^+ \) from \((0, 0)\) to \((x, y)\).]
II. Combination of solutions by H.L. Abbott, University of Alberta, and by Chris Wildhagen, Breda, The Netherlands.

Let the half-plane be given by \( \{(x, y) : y \geq 0\} \) and let \( g_k(n) \) denote the number of walks with \( n \) steps which have initial point \((0, k)\) and which do not leave the half-plane. The problem calls for \( g_0(n) \). Observe that for \( k \geq 1 \),

\[
g_k(n) = 2g_k(n \leftrightarrow 1) + g_{k+1}(n \leftrightarrow 1) + g_{k-1}(n \leftrightarrow 1).
\]  

(3)

In fact, (3) follows from the observation that the walks counted by \( g_k(n) \) may be split into four sets according to the direction of the first step. Similarly, one gets

\[
g_0(n) = 2g_0(n \leftrightarrow 1) + g_1(n \leftrightarrow 1).
\]  

(4)

We have the boundary conditions

\[
g_0(1) = 3 \quad \text{and} \quad g_k(n) = 4^n \quad \text{for} \quad k \geq n.
\]  

(5)

It is clear that \( g_k(n) \) is uniquely determined by (3), (4) and (5). We shall prove by induction on \( n \) that

\[
g_k(n) = \sum_{j=0}^{k} \binom{2n+1}{n \leftrightarrow j}
\]

satisfies (3), (4) and (5). \( g_0(1) = 3 \) is obvious, and for \( k \geq n \)

\[
\sum_{j=1}^{k} \binom{2n+1}{n \leftrightarrow j} = \sum_{i=0}^{n} \binom{2n+1}{i} = \frac{1}{2} \sum_{i=0}^{2n+1} \binom{2n+1}{i} = \frac{1}{2} 2^{2n+1} = 4^n,
\]

so (5) holds. For (4) we have

\[
g_0(n) = 2 \left( \binom{2n \leftrightarrow 1}{n \leftrightarrow 1} + \binom{2n \leftrightarrow 1}{n \leftrightarrow 1} + \binom{2n \leftrightarrow 1}{n \leftrightarrow 1} \right)

= \left( \binom{2n \leftrightarrow 1}{n \leftrightarrow 1} \right) + 2 \left( \binom{2n \leftrightarrow 1}{n \leftrightarrow 1} + \binom{2n \leftrightarrow 1}{n \leftrightarrow 1} \right)

= \left( \binom{2n}{n} \right) + \binom{2n}{n \leftrightarrow 1} = \binom{2n+1}{n}.
\]

and for (3)

\[
g_k(n) = 2 \sum_{j=0}^{k} \binom{2n \leftrightarrow 1}{n \leftrightarrow 1 \leftrightarrow j} + \sum_{j=0}^{k+1} \binom{2n \leftrightarrow 1}{n \leftrightarrow 1 \leftrightarrow j} + \sum_{j=0}^{k-1} \binom{2n \leftrightarrow 1}{n \leftrightarrow 1 \leftrightarrow j}

= 2 \sum_{j=0}^{k} \binom{2n \leftrightarrow 1}{n \leftrightarrow 1 \leftrightarrow j} + \sum_{j=0}^{k} \binom{2n \leftrightarrow 1}{n \leftrightarrow 2 \leftrightarrow j} + \binom{2n \leftrightarrow 1}{n \leftrightarrow 1} + \sum_{j=0}^{k} \binom{2n \leftrightarrow 1}{n \leftrightarrow j} 

= \sum_{j=0}^{k} \left[ \binom{2n \leftrightarrow 1}{n \leftrightarrow 1 \leftrightarrow j} + \binom{2n \leftrightarrow 1}{n \leftrightarrow 2 \leftrightarrow j} \right] + \sum_{j=0}^{k} \left[ \binom{2n \leftrightarrow 1}{n \leftrightarrow 1 \leftrightarrow j} + \binom{2n \leftrightarrow 1}{n \leftrightarrow j} \right]

= \sum_{j=0}^{k} \binom{2n}{n \leftrightarrow 1 \leftrightarrow j} + \sum_{j=0}^{k} \binom{2n}{n \leftrightarrow j} = \sum_{j=0}^{k} \binom{2n+1}{n \leftrightarrow j}.
\]
In particular, the required number of walks is

\[ g_0(n) = \binom{2n+1}{n}. \]

Also solved by W. BRECKENRIDGE (student), H. GASTINEAU-HILLS, A. NELSON, P. BOS (student), G. CALVERT (student) and K. WEHRHAHN, all of the University of Sydney, Australia; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD K. GUY, University of Calgary; GEORGE P. HENDERSON, Campbellcroft, Ontario; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; CHRISTIAN KRATTENTHALER, Institut für Mathematik, Vienna, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; BRUCE E. SAGAN, Michigan State University, East Lansing; ROBERT E. SHAFER, Berkeley, California; and GEORGE SZEKERES, University of New South Wales, Kensington, Australia. Two other readers sent in the correct answer, one with no proof and one with an incorrect proof.

Breckenridge et al have written an article, “Lattice paths and Catalan numbers”, which has appeared in the Bulletin of the Institute of Combinatorics and its Applications, Vol. 1 (1991), pp. 41-55, dealing with this problem and related questions. In this paper they give a proof of the present problem which is very similar to solution I.

Guy, Krattenthaler and Sagan have also written a paper, “Lattice paths, reflections, and dimension-changing bijections” (submitted for publication), in which they solve this problem and tackle several analogous problems in the plane and in higher dimensions. Guy has compiled separately an extensive annotated list of references pertaining to related topics.

To the editor’s surprise, no reference to the precise problem asked could be supplied by the readers.

* * * * *


O and H are respectively the circumcenter and orthocenter of a triangle ABC in which \( \angle A \neq 90^\circ \). Characterize triangles ABC for which \( \Delta AOH \) is isosceles. Which of these triangles ABC have integer sides?

Combined solutions of Marcin E. Kuczma, Warszawa, Poland; and the proposer.

Triangle \( \Delta AOH \) can be isosceles in three ways. In each case, the reasoning is written in the form of successively equivalent statements; the first line identifies the case and the last line gives the desired characterization of triangle ABC. The vector equality

\[ \overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \]

is the clue. \( R \) denotes the circumradius, and sums are cyclic over \( A, B, C \).

Case (i): \( AH = AO \)

\[ \Leftrightarrow |\overrightarrow{OH}| = |\overrightarrow{OA}| \Leftrightarrow (\overrightarrow{OB} + \overrightarrow{OC})^2 = (\overrightarrow{OA})^2 \]

\[ \Leftrightarrow 2R^2 + 2 \overrightarrow{OB} \cdot \overrightarrow{OC} = R^2 \Leftrightarrow 2 \cos 2A = \Leftrightarrow 1 \]

\[ \Leftrightarrow \angle A = 60^\circ \text{ or } 120^\circ. \]
Case (ii): $OH = OA$
\[ \Leftrightarrow (OA + OB + OC)^2 = (OA)^2 \Leftrightarrow 3R^2 + 2\sum OB \cdot OC = R^2 \]
\[ \Leftrightarrow \sum \cos 2a = \Leftrightarrow \cos A \cos B \cos C = 0 \]
[using $4 \cos A \cos B \cos C = 1 + \sum \cos 2A$]
\[ \Leftrightarrow \angle B = 90^\circ \text{ or } \angle C = 90^\circ. \]

Case (iii): $HO = HA$
\[ \Leftrightarrow |OH| = |OH \Leftrightarrow OA| \Leftrightarrow (OA + OB + OC)^2 = (OB + OC)^2 \]
\[ \Leftrightarrow 3R^2 + 2\sum OB \cdot OC = 2R^2 + 2 \cdot OB \cdot OC \]
\[ \Leftrightarrow 2(\cos 2B + \cos 2C) = \Leftrightarrow 1 \]
\[ \Leftrightarrow 4 \cos A \cos(B \Leftrightarrow C) = 1 \]
[using $\cos X + \cos Y = 2 \cos \left(\frac{X + Y}{2}\right) \cos \left(\frac{X - Y}{2}\right)$ with $X = 2B$, $Y = 2C$.] This solves the first part of the problem.

To cope with the last question, we must restate the resultant equalities in terms of side lengths. In case (i) we are led to the alternatives
\[ a^2 = b^2 \Leftrightarrow bc + c^2 \quad \text{or} \quad a^2 = b^2 + bc + c^2; \quad (1) \]
in case (ii) to
\[ b^2 = a^2 + c^2 \quad \text{or} \quad c^2 = a^2 + b^2. \quad (2) \]

If $a^2 = b^2 \Leftrightarrow bc + c^2$ in (1), without loss of generality let $b > c$. Then $a^2 \Leftrightarrow c^2 = b^2 \Leftrightarrow bc$
so that
\[ \frac{a \Leftrightarrow c}{b \Leftrightarrow c} = \frac{b}{a + c} = \frac{m}{n} \]
in lowest terms. Thus
\[ a \Leftrightarrow c = \frac{m}{n} (b \Leftrightarrow c) \quad \text{or} \quad a + \left(\frac{m \Leftrightarrow n}{n}\right) c = \left(\frac{m}{n}\right) b \]
and
\[ a + c = \left(\frac{n}{m}\right) b. \]
Solving for $a$ and $c$, we have
\[ a = \left(\frac{m^2 \Leftrightarrow mn + n^2}{2mn \Leftrightarrow m^2}\right) b, \quad c = \left(\frac{n^2 \Leftrightarrow m^2}{2mn \Leftrightarrow m^2}\right) b. \]

Thus
\[ a = k(m^2 \Leftrightarrow mn + n^2), \quad b = k(2mn \Leftrightarrow m^2), \quad c = k(n^2 \Leftrightarrow m^2). \quad (3) \]
Similarly from $a^2 = b^2 + bc + c^3$ we obtain
\[ a = k(m^2 + mn + n^2), \quad b = k(2mn + m^2), \quad c = k(n^2 \Leftrightarrow m^2). \quad (4) \]
As \( m, n \) range over all pairs of relatively prime integers with \( n > m > 0 \), the sets of formulas in (3) and (4) for \( k = 1 \) produce triples of positive integers \( a, b, c \) which are either mutually coprime or divisible by 3; after possible reduction by 3 we obtain the complete solution to the equations in (1) in irreducible triples of positive integers.

For (2) the irreducible solutions are the Pythagorean triples

\[
a = 2xy, \quad b = x^2 + y^2, \quad c = x^2 \Leftrightarrow y^2
\]

or

\[
a = 2xy, \quad b = x^2 \Leftrightarrow y^2, \quad c = x^2 + y^2,
\]

with a possible reduction factor of 2.

For case (iii) we have

\[
4 \cos A \cos(B \Leftrightarrow C) = 1 \Leftrightarrow 4 \cos(B + C) \cos(B \Leftrightarrow C) = \Leftrightarrow 1
\]

\[
\Leftrightarrow 2(\cos 2B + \cos 2C) = \Leftrightarrow 1
\]

\[
\Leftrightarrow 4 \cos^2 B + 4 \cos^2 C = 3. \quad (5)
\]

By the cosine formula, \( 2 \cos B \) and \( 2 \cos C \) are rational in a triangle with integer sides, so (5) can be written in the form

\[
x^2 + y^2 = 3z^2
\]

for \( x, y, z \) integral and not all even, which, by reducing modulo 4, clearly has no solution. Thus there are no solutions in case (iii).

There were two partial solutions submitted.

The proposer’s original problem actually contained the additional condition that the area of the triangle also be integral; as the proposer points out, case (i) above then has no solution, since \( \sin 60^\circ \) is irrational, and so the only solutions are the right-angled triangles of case (ii). The editor, however, inadvertently omitted the area condition from the problem.

* * * * *


Find all prime numbers which, written in the number system with base \( b \), contain each digit \( 0, 1, \ldots, b \Leftrightarrow 1 \) exactly once (a leading zero is allowed).

Solution by Richard I. Hess, Rancho Palos Verdes, California.

If \( p \) is such a prime number, with base \( b \) representation \((a_{b-1} a_{b-2} \cdots a_1 a_0)_b\), then

\[
p = a_0 + a_1 b + a_2 b^2 + \cdots + a_{b-1} b^{b-1}
= (a_0 + a_1 + \cdots + a_{b-1}) + a_1 (b \Leftrightarrow 1) + a_2 (b^2 \Leftrightarrow 1) + \cdots + a_{b-1} (b^{b-1} \Leftrightarrow 1)
= \frac{b(b \Leftrightarrow 1)}{2} + a_1 (b \Leftrightarrow 1) + a_2 (b^2 \Leftrightarrow 1) + \cdots + a_{b-1} (b^{b-1} \Leftrightarrow 1).
\]
Thus for $b = 2k$, $p$ will be divisible by $b = 1$, so the only such primes for even bases are for $b = 2$; in fact, only $2 = (10)_2$. For $b = 2k + 1$, $p$ will be divisible by $k$, so the only such primes for odd bases are for $b = 3$, and we find the solutions $(012)_3 = 5, (021)_3 = 7, (102)_3 = 11$, and $(201)_3 = 19$.

Also solved by H.L. ABBOTT, University of Alberta; DUANE M. BROLINE, Eastern Illinois University, Charleston; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; T. LEINSTER, Lancing College, England; R.E. SHAFER, Berkeley, California; ST. OLAF PROBLEM SOLVING GROUP, St. Olaf College, Northfield, Minnesota; CHRIS WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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A point is said to be inside a parabola if it is on the same side of the parabola as the focus. Given a finite number of parabolas in the plane, must there be some point of the plane that is not inside any of the parabolas?

**Solution by Jordi Dou, Barcelona, Spain.**

Let $\ell$ be a line not parallel to any of the axes of the given parabolas $P_1, \ldots, P_n$.

The interior points of $P_i$ that belong to $\ell$, if there are any, form a segment $I_i = A_iB_i$ where $\ell \cap P_i = \{A_i, B_i\}$. The set of points of $\ell$ interior to any of the $n$ parabolas is then the union $\bigcup_{i=1}^n I_i$, which consists of at most $n$ segments of finite total length, while the remainder, points exterior to all the parabolas $P_i$, have infinite length.

Another proof is as follows. Given $n$ parabolas and a point $O$, let $\Omega$ be a circle of centre $O$ and variable radius $x$. Denote by $I(x)$ the area of the region, contained in $\Omega(x)$, formed by points interior to any parabola, and by $E(x)$ the area of the complementary region $\Omega(x) \not\subset I(x)$, formed by points exterior to all the parabolas. So $I(x) + E(x) = \pi x^2$. Then we can prove that $\lim_{x \to \infty} (I(x)/E(x)) = 0$. (The reason is that $I(x)$ is of order $x\sqrt{x}$, while $E(x)$ is of order $x^2$.)

Also proved (usually by one or the other of the above methods) by H.L. ABBOTT, University of Alberta; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; T. LEINSTER, Lancing College, England; R.C. LYNNESS, Southwold, England; P. PENNING, Delft, The Netherlands; DAVID SINGMASTER, South Bank Polytechnic, London, England; ST. OLAF PROBLEM SOLVING GROUP, St. Olaf College, Northfield, Minnesota; and the proposer. A comment was also received from a reader who evidently didn’t understand the problem.

One of the above solvers (whose identity will be kept secret) claims that the same result is true for a countable number of parabolas. Is it?
This is the third solution of Jordi Dou to be featured in this issue. It seems fitting to disclose here that Dou recently wrote to the editor, and (while conveying 80th birthday greetings to Dan Pedoe) mentioned that he had himself just turned 80 on June 7! Crux readers, and certainly this admiring and appreciative editor, would wish Professor Dou all the best.

* * * * *

Triangle \(ABC\) has angles \(\alpha, \beta, \gamma\), circumcenter \(O\), incenter \(I\), and orthocenter \(H\). Suppose that the points \(A, H, I, O, B\) are concyclic.

(a) Find \(\gamma\).
(b) Prove \(HI = IO\).
(c) If \(AH = HI\), find \(\alpha\) and \(\beta\).

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

(a) If \(A, H, I, O, B\) are concyclic, then

\[\angle AHB = \angle AIB = \angle AOB,\]

so

\[180^\circ \iff \gamma = 90^\circ + \frac{\gamma}{2} = 2\gamma,\]

so \(\gamma = 60^\circ\).

Remark: if two of the three points \(H, I, O\) are concyclic with \(A\) and \(B\), then the third one is as well. [Editor’s note. This assumes \(\triangle ABC\) is acute. If \(\angle C = 120^\circ\) then \(A, H, O, B\) are concyclic but \(I\) does not lie on the same circle.]

(b) From \(\angle ABH = 90^\circ \iff \alpha = \angle OBC\) we obtain

\[\angle OBI = \frac{1}{2}(\alpha \iff \gamma) = \angle IBH,\]

so \(HI = IO\).

(c) If \(AH = HI\) then

\[90^\circ \iff \alpha = \angle ABH = \angle HBI = \frac{\alpha}{2} \iff \frac{\gamma}{2} = \frac{\alpha}{2} \iff 30^\circ,\]

so \(\alpha = 80^\circ\) and thus \(\beta = 40^\circ\).

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, and JOSÉ YUSTY PITA, Madrid, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; HIDETOSI FUKAGAWA, Aichi, Japan; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong;
TOM LEINSTER, Lancing College, England; P. PENNING, Delft, The Netherlands; K.R.S. SASTRY, Addis Ababa, Ethiopia; TOSHI SEIMIYA, Kawasaki, Japan; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

Engelhaupt, Fukagawa and Kuczma observed that it need only be assumed that $A, I, O, B$ are concyclic, Leinster the same when $A, H, I, B$ are concyclic. Dou and Sastry give converses. Dou also notes that under the hypothesis of the problem, segments $OH$ and $CO'$ are perpendicular bisectors of each other, where $O'$ is the centre of circle $ABOIH$. The proposer also proved that, letting $S$ be the second intersection of $BC$ with circle $ABOIH$, $IS$ is parallel to $AB$ and $IS = AI = SB$.

* * * * *

Show that if $a, b, c, d, x, y > 0$ and

$$xy = ac + bd, \quad \frac{x}{y} = \frac{ad + bc}{ab + cd},$$

then

$$\frac{abx}{a + b + x} + \frac{cdx}{c + d + x} = \frac{ady}{a + d + y} + \frac{bcy}{b + c + y}.$$

Note that, from the given expressions for $xy$ and $x/y$,

$$ab(c + d + x) + cd(a + b + x) = ad(b + c + y) + bc(a + d + y)$$

and

$$x(a + d + y)(b + c + y) = ((a + d)x + ac + bd)(b + c + y)$$
$$= (ac + bd)y + (a + d)(b + c)x + (b + c)(ac + bd) + (a + d)(ac + bd)$$
$$= (ac + bd)x + (a + b)(c + d)y + (c + d)(ac + bd) + (a + b)(ac + bd)$$
$$= ((a + b)y + ac + bd)(c + d + x)$$
$$= y(a + b + x)(c + d + x).$$

It follows that

$$x \left( \frac{ab}{a + b + x} + \frac{cd}{c + d + x} \right) = y \left( \frac{ad}{a + d + y} + \frac{bc}{b + c + y} \right).$$
II. Combination of partial solutions by Wilson da Costa Areias, Rio de Janeiro, Brazil, and Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

It is well known (see p. 111, no. 207 of N. Altshiller-Court, *College Geometry*) that it is possible to construct a cyclic quadrilateral $ABCD$ of sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ [provided that $a < b + c + d$, etc. — Ed.] and also that, from Ptolemy’s theorem, its diagonals $x = AC$, $y = BD$ satisfy

$$xy = ac + bd, \quad \frac{x}{y} = \frac{ad + bc}{ab + cd}.$$  

Let $R$ be the circumradius of $ABCD$, and let $F_1, F_2, F_3, F_4$ and $s_1, s_2, s_3, s_4$ be the areas and semiperimeters of the triangles $ABC$, $BCD$, $CDA$, $DAB$, respectively. Then

$$\frac{abx}{a + b + x} + \frac{cdx}{c + d + x} = \frac{bcy}{b + c + y} + \frac{ady}{a + d + y}$$

is equivalent to

$$\frac{F_1 \cdot 4R}{2s_1} + \frac{F_3 \cdot 4R}{2s_3} = \frac{F_2 \cdot 4R}{2s_2} + \frac{F_4 \cdot 4R}{2s_4},$$

or

$$r_1 + r_3 = \frac{F_1}{s_1} + \frac{F_3}{s_3} = \frac{F_2}{s_2} + \frac{F_4}{s_4} = r_2 + r_4,$$

where $r_1, r_2, r_3, r_4$ are respectively the inradii of the above triangles. The relation $r_1 + r_3 = r_2 + r_4$ is true and has been shown by H. Forder in “An ancient Chinese theorem”, Math Note 2128, p. 68 of *Mathematical Gazette* 34, no. 307 (1950). It was also part (b) of *Crux* 1226 [1988: 147].

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; GEORGE P. HENDERSON, Campbelcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; TOM LEINSTER, Lancing College, England; J.A. MCCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Penn State University at Harrisburg; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Murty points out that the result holds for all real $a, b, c, d, x, y$ provided only that the denominators in the problem are nonzero (as is clear from solution I).

The geometrical interpretation given in (partial) solution II was also noted by Festraets-Hamoir, Kuczma, Smeenk and the proposer. Most of them, however, gave algebraic proofs, thus avoiding the cases (e.g., $a > b + c + d$) missed by Areias and Bellot. Kuczma gave a separate and rather involved argument to cover these cases. Is there a “nice” way to complete solution II?

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The Olympiad Corner: No. 125 ..............................  R.E. Woodrow  129

Problems: 1641–1650 ......................................................  140

Solutions: 875, 1523–1528, 1530, 1531 ..............................  141
All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month with the problems of the 40th Mathematical Olympiad from Poland. This was written in April 1989. My thanks to Marcin E. Kuczma for sending the problem set to me.

40th MATHEMATICAL OLYMPIAD IN POLAND

Final round (April, 1989)

1. An even number of people are participating in a round table conference. After lunch break the participants change seats. Show that some two persons are separated by the same number of persons as they were before break.

2. $K_1, K_2, K_3$ are pairwise externally tangent circles in the plane. $K_2$ touches $K_3$ at $P$, $K_3$ touches $K_1$ at $Q$, $K_1$ touches $K_2$ at $R$. Lines $PQ$ and $PR$ cut $K_1$ in the respective points $S$ and $T$ (other than $Q$ and $R$). Lines $SR$ and $TQ$ cut $K_2$ and $K_3$ in the respective points $U$ and $V$ (other than $R$ and $Q$). Prove that $P$ lies in line with $U$ and $V$.

3. The edges of a cube are numbered 1 through 12.

   (a) Prove that for every such numbering there exist at least eight triples of integers $(i, j, k)$ with $1 \leq i < j < k \leq 12$ such that the edges assigned numbers $i, j, k$ are consecutive segments of a polygonal line.

   (b) Give an example of a numbering for which a ninth triple with properties stated in (a) does not exist.

4. Let $n$ and $k$ be given positive integers. Consider a chain of sets $A_0, \ldots, A_k$ in which $A_0 = \{1, \ldots, n\}$ and, for each $i$ ($i = 1, \ldots, k$), $A_i$ is a randomly chosen subset of $A_{i-1}$; all choices are equiprobable. Show that the expected cardinality of $A_k$ is $n2^{-k}$.

5. Three pairwise tangent circles of equal radius $a$ lie on a hemisphere of radius $r$. Determine the radius of a fourth circle contained in the same sphere and tangent to the three given ones.

6. Prove that the inequality

$$\left(\frac{ab + ac + ad + bc + bd + cd}{6}\right)^{1/2} \geq \left(\frac{abc + abd + acd + bcd}{4}\right)^{1/3}$$

holds for any positive numbers $a, b, c, d$.

* * *

A second Olympiad for this issue comes from Sweden via Andy Liu.
SWEDISH MATHEMATICAL COMPETITION

Final round: November 18, 1989
Time: 5 hours

1. Let \( n \) be a positive integer. Prove that the integers \( n^2(n^2 + 2)^2 \) and \( n^4(n^2 + 2)^2 \) can be written in base \( n^2 + 1 \) with the same digits but in opposite order.

2. Determine all continuous functions \( f \) such that \( f(x) + f(x^2) = 0 \) for all real numbers \( x \).

3. Find all positive integers \( n \) such that \( n^3 - 18n^2 + 115n - 391 \) is the cube of a positive integer.

4. Let \( ABCD \) be a regular tetrahedron. Where on the edge \( BD \) is the point \( P \) situated if the edge \( CD \) is tangent to the sphere with diameter \( AP \)?

5. Assume that \( x_1, \ldots, x_5 \) are positive real numbers such that \( x_1 < x_2 \) and assume that \( x_3, x_4, x_5 \) are all greater than \( x_2 \). Prove that if \( \alpha > 0 \), then
\[
\frac{1}{(x_1 + x_3)\alpha} + \frac{1}{(x_2 + x_4)\alpha} + \frac{1}{(x_2 + x_5)\alpha} < \frac{1}{(x_1 + x_2)\alpha} + \frac{1}{(x_2 + x_3)\alpha} + \frac{1}{(x_4 + x_5)\alpha}.
\]

6. On a circle \( 4n \) points, \( n \geq 1 \), are chosen. Every second point is colored yellow, the other points are colored blue. The yellow points are divided into \( n \) pairs and the points in each pair are connected with a yellow line segment. In the same manner the blue points are divided into \( n \) pairs and the points in each pair are connected with a blue line segment. Assume that at most two line segments pass through each point in the interior of the circle. Prove that there are at least \( n \) points of intersection of blue and yellow line segments.

* 

It has become a custom to give the problems of the Asian Pacific Mathematics Olympiad. Since “official solutions” for this contest are widely distributed we will only publish particularly novel and interesting solutions. My thanks to Ed Barbeau and Andy Liu for sending me this problem set.

1991 ASIAN PACIFIC MATHEMATICS OLYMPIAD

March 1991
Time allowed: 4 hours

1. Given \( \triangle ABC \), let \( G \) be the centroid and \( M \) be the mid-point of \( BC \). Let \( X \) be on \( AB \) and \( Y \) on \( AC \) such that the points \( X, G \) and \( Y \) are collinear and \( XGY \) and \( BC \) are parallel. Suppose that \( XC \) and \( GB \) intersect at \( Q \) and that \( YB \) and \( GC \) intersect at \( P \). Show that \( \triangle MPQ \) is similar to \( \triangle ABC \).
2. Suppose there are 997 points given on a plane. If every two points are joined by a line segment with its mid-point coloured in red, show that there are at least 1991 red points on the plane. Can you find a special case with exactly 1991 red points?

3. Let \(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\) be positive real numbers such that

\[
\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k.
\]

Show that

\[
\sum_{k=1}^{n} \frac{(a_k)^2}{a_k + b_k} \geq \frac{1}{2} \sum_{k=1}^{n} a_k.
\]

4. During a break \(n\) children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule: he selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of \(n\) for which eventually (perhaps after many rounds) all children will have at least one candy each.

5. Given are two tangent circles, \(C_1, C_2\), and a point \(P\) on their radical axis, i.e. on the common tangent of \(C_1\) and \(C_2\) that is perpendicular to the line joining the centres of \(C_1\) and \(C_2\). Construct with compass and ruler all the circles \(C\) that are tangent to \(C_1\) and \(C_2\) and pass through the point \(P\).

*  

I am giving more problems than usual this issue to give readers some sources of pleasure for the summer break. Also the June issue is normally taken up by two contests for which we do not usually publish readers’ solutions.

*  

Last issue we gave the problems of the A.I.M.E. for 1991. As promised, we next give the numerical solutions. The problems and their official solutions are copyrighted by the Committee of the American Mathematics Competitions of the Mathematical Association of America, and may not be reproduced without permission. Detailed solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, U.S.A., 68588-0322.

1.  146  
2.  840  
3.  166  
4.  159  
5.  128  
6.  743  
7.  383  
8.  010  
9.  044  
10.  532  
11.  135  
12.  677  
13.  990  
14.  384  
15.  012  

*  

We now turn to some further solutions submitted by readers for problems from the ‘Archives’. First a problem from April 1984.
Prove that, for all natural numbers \( n \geq 2 \),
\[
\prod_{i=1}^{n} \tan \left( \frac{\pi}{3} \left( 1 + \frac{3^i}{3^n - 1} \right) \right) = \prod_{i=1}^{n} \cot \left( \frac{\pi}{3} \left( 1 - \frac{3^i}{3^n - 1} \right) \right).
\]

Solution by Murray S. Klamkin, University of Alberta.
Let
\[
A_i = \tan \left( \frac{\pi}{3} \left( 1 + \frac{3^i}{3^n - 1} \right) \right) = \frac{\tan \frac{\pi}{3} + \tan \left( \frac{3^{i-1}}{3^n - 1} \right)}{1 - \tan \frac{\pi}{3} \tan \left( \frac{3^{i-1}}{3^n - 1} \right)}
\]
and
\[
B_i = \tan \left( \frac{\pi}{3} \left( 1 - \frac{3^i}{3^n - 1} \right) \right) = \frac{\tan \frac{\pi}{3} - \tan \left( \frac{3^{i-1}}{3^n - 1} \right)}{1 + \tan \frac{\pi}{3} \tan \left( \frac{3^{i-1}}{3^n - 1} \right)}.
\]
Now since
\[
\tan 3\theta = \tan \theta \left[ \frac{3 - \tan^2 \theta}{1 - 3 \tan^2 \theta} \right]
\]
we get
\[
A_i B_i = \frac{3 - \tan^2 \left( \frac{3^{i-1}}{3^n - 1} \right)}{1 - 3 \tan^2 \left( \frac{3^{i-1}}{3^n - 1} \right)} = \frac{\tan \left( \frac{3^i}{3^n - 1} \right)}{\tan \left( \frac{3^{i-1}}{3^n - 1} \right)}.
\]
Hence
\[
\prod_{i=1}^{n} A_i B_i = \frac{\tan \left( \frac{3^n}{3^n - 1} \right)}{\tan \left( \frac{3}{3^n - 1} \right)} = \frac{\tan \left( \frac{\pi}{3} + \frac{\pi}{3^n - 1} \right)}{\tan \left( \frac{\pi}{3^n - 1} \right)} = 1.
\]
From this the result is immediate.

* 

Determine all real solutions \( x, y \) of the system
\[
\begin{align*}
\quad x^4 + y^2 - xy^3 - 9x/8 &= 0, \\
y^4 + x^2 - yx^3 - 9y/8 &= 0.
\end{align*}
\]

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.
We find instead all the solutions of the given equations, which we rewrite as
\[
\begin{align*}
8x^4 + 8y^2 - 8xy^3 - 9x &= 0, & (1) \\
8y^4 + 8x^2 - 8yx^3 - 9y &= 0. & (2)
\end{align*}
\]
We show there are exactly ten real or complex solutions, namely
\[
(0, 0), \left( \frac{9}{8}, \frac{9}{8} \right), \left( 1, \frac{1}{2} \right), \left( \frac{1}{2}, 1 \right), \left( \omega, \frac{\omega^2}{2} \right), \left( \omega^2, \frac{\omega}{2} \right), \left( \frac{\omega^2}{2}, \omega \right), \left( \frac{\omega}{2}, \omega^2 \right), \left( \frac{9\omega}{8}, \frac{9\omega^2}{8} \right), \left( \frac{9\omega^2}{8}, \frac{9\omega}{8} \right),
\]

From this the result is immediate.
where \( \omega = (-1 + \sqrt{3}i)/2 \) denotes a complex cube root of unity.

From \((y \text{ times } (1)) \text{ minus } (x \text{ times } (2))\) we get
\[
0 = 2(x^4y - xy^4) - (x^3 - y^3) = (x^3 - y^3)(2xy - 1)
= (x - y)(2xy - 1)(x^2 + xy + y^2) .
\]

If \( x = y \), then substitution in \((1)\) yields \( 8x^2 - 9x = 0 \) which immediately gives two candidates \( (0, 0) \) and \( (9/8, 9/8) \). If \( 2xy = 1 \), then substituting \( y = (2x)^{-1} \) in \((1)\) and simplifying we obtain
\[
8x^4 + \frac{1}{x^2} - 9x = 0
\]
or \( 8x^6 - 9x^3 + 1 = 0 \). Setting \( t = x^3 \) we obtain \( 8t^2 - 9t + 1 = 0 \) which yields \( t = 1, 1/8 \). These give six more candidates
\[
(1, \frac{1}{2}), (\omega, \frac{\omega^2}{2}), (\omega^2, \frac{\omega}{2}), (\frac{1}{2}, 1), (\frac{\omega}{2}, \omega^2), (\frac{\omega^2}{2}, \omega).
\]

Finally, suppose
\[
x^2 + xy + y^2 = 0 . \tag{3}
\]

From \((1) \text{ minus } (2)\) we obtain
\[
8(x^4 - y^4) - 8(x^2 - y^2) + 8xy(x^2 - y^2) - 9(x - y) = 0 ,
\]
and disregarding the possibility that \( x = y \), a case already considered above, we have
\[
(x + y)[8(x^2 + y^2) - 8 + 8xy] = 9 . \tag{4}
\]

Substitution of \((3)\) into \((4)\) now yields
\[
x + y = -\frac{9}{8} . \tag{5}
\]

From \((5) \text{ and } (3)\) we get
\[
xy = (x + y)^2 - (x^2 + xy + y^2) = \frac{81}{64} . \tag{6}
\]

From \((5) \text{ and } (6)\) we see that \( x \) and \( y \) are the two roots of the equation
\[
u^2 + \frac{9}{8}u + \frac{81}{64} = 0 ,
\]
or \( 64u^2 + 72u + 81 = 0 \). Solving, we get \( u = (-9 \pm 9\sqrt{3}i)/16 \). This gives two more (complex) candidates
\[
\left(\frac{9\omega}{8}, \frac{9\omega^2}{8}\right), \left(\frac{9\omega^2}{8}, \frac{9\omega}{8}\right).
\]

Straightforward substitution into \((1) \text{ and } (2)\) shows that all of these ten candidates are solutions.
**Editor’s note.** Solutions were also received from Nicos Diamantis, student, University of Patras, Greece, and Hans Engelhaupt, Gundelsheim, Germany.


The consecutive vertices of a given convex n-gon are $A_0, A_1, \ldots, A_{n-1}$. The n-gon is partitioned into $n - 2$ triangles by diagonals which are non-intersecting (except possibly at the vertices). Show that there exists an enumeration $\Delta_1, \Delta_2, \ldots, \Delta_{n-2}$ of these triangles such that $A_i$ is a vertex of $\Delta_i$ for $1 \leq i \leq n - 2$. How many enumerations of this kind exist?

**Solution by Hans Engelhaupt, Gundelsheim, Germany.**

There is always just one such enumeration.

We argue by induction on $n$. The cases $n = 3$ and $n = 4$ are trivial. Suppose $n > 4$.

The side $A_0A_{n-1}$ and another vertex, say $A_k$, $1 \leq k \leq n - 2$, form a triangle. It is immediate that the triangle is $\Delta_k$ (since 0 and $n - 2$ are not available labels). If $k = 1$ or $k = n - 2$, by considering the remaining $(n - 1)$-gon formed using the diagonal $A_{n-1}A_1$ or $A_0A_{n-2}$ as appropriate and relabelling $A_i$ as $A'_{i-1}$ in the former case, the result follows.

So suppose $1 < k < n - 2$. Now the triangle formed divides the polygon into two convex polygons $[A_0, \ldots, A_k]$ and $[A_k, \ldots, A_{n-1}]$. Also the original triangulation induces a triangulation of each of these, since the diagonals do not intersect except at endpoints. Existence of a numbering is now immediate. Uniqueness follows since the triangles $\Delta_1, \ldots, \Delta_{k-1}$ must be in $[A_0, \ldots, A_k]$ and $\Delta_{k+1}, \ldots, \Delta_{n-2}$ must be in $[A_k, \ldots, A_{n-1}]$.

* 

We now turn to the March 1986 numbers, and solutions to some of the problems of the 1982 Leningrad High School Olympiad (Third Round) [1986: 39-40].

1. $P_1$, $P_2$ and $P_3$ are quadratic trinomials with positive leading coefficients and real roots. Show that if each pair of them has a common root, then the trinomial $P_1 + P_2 + P_3$ also has real roots. (Grade 8)
Solution by Hans Engelhaupt, Gundelsheim, Germany.

Let the trinomials be

\[ P_1 : ax^2 + bx + c = 0 \text{ with } a > 0 \text{ and the real roots } u_1, u_2 ; \]
\[ P_2 : dx^2 + ex + f = 0 \text{ with } d > 0 \text{ and the real roots } v_1, v_2 ; \]
and
\[ P_3 : gx^2 + hx + i = 0 \text{ with } g > 0 \text{ and the real roots } w_1, w_2 . \]

Without loss of generality \( u_1 = v_1 \).

Case 1: \( u_1 = w_1 \) or \( u_1 = w_2 \).

Then \( P_1 + P_2 + P_3 \) has the real root \( u_1 \) so the other root is real as well.

Case 2: \( u_2 = w_1 \) and \( v_2 = w_2 \).

Then

\[ P_1 + P_2 + P_3 = a(x - u_1)(x - u_2) + d(x - u_1)(x - v_2) + g(x - u_2)(x - v_2) \ (= f(x)). \]

By renumbering \( P_1, P_2, P_3 \) and the roots if necessary, we may assume without loss of generality that \( u_1 < u_2 < v_2 \). Now \( f(u_1) > 0, f(u_2) < 0 \) and \( f(v_2) > 0 \). By the intermediate value theorem \( f(x) \) has two real roots \( z_1, z_2 \) with

\[ u_1 < z_1 < u_2 < z_2 < v_2. \]

By a suitable renumbering, any situation reduces to Case 1 or Case 2, completing the proof.

2. If in triangle \( ABC \), \( C = 2A \) and \( AC = 2BC \), show that it is a right triangle. (Grade 8, 9)

Solution by Hans Engelhaupt, Gundelsheim, Germany.

Choose \( D \) on the line \( BC \) (extended) so that \( CD = AC \). Then \( BD = 3BC \). The triangles \( ADB \) and \( CAB \) are similar because \( \angle CAD = \angle ADB = A \) (= \( \alpha \), say). Thus \( \frac{AB}{BD} = \frac{BC}{AB} \) and \( AB^2 = BD \cdot BC = 3 \cdot BC^2 \). Therefore in triangle \( ABC \), \( AB^2 + BC^2 = AC^2 \), and \( B = 90^\circ \), as desired.

Editor’s note. A solution using the law of sines was sent in by Bob Prielipp, University of Wisconsin–Oshkosh.

8. Prove that for any natural number \( k \), there is an integer \( n \) such that

\[ \sqrt{n + 1981^k} + \sqrt{n} = (\sqrt{1982} + 1)^k. \]

(Grade 9)
Solution by Bob Prielipp, University of Wisconsin–Oshkosh.

Let

\[ A = \sum_{j \text{ even}}^k \binom{k}{j} (\sqrt{1982})^j, \quad B = \sum_{j \text{ odd}}^k \binom{k}{j} (\sqrt{1982})^j. \]

[Thus \( A \) is the sum of the even-numbered terms in the expansion of \((\sqrt{1982} + 1)^k\) and \( B \) is the sum of the odd-numbered terms in that expansion.] Note also that

\[ (\sqrt{1982} - 1)^k = \sum_{j=0}^{k} \binom{k}{j} (\sqrt{1982})^{k-j}(-1)^j = \begin{cases} B - A & \text{if } k \text{ is odd} \\ A - B & \text{if } k \text{ is even} \end{cases}. \]

**Case 1:** \( k \) is odd. Let \( n = A^2 \). Then

\[
\sqrt{n + 1981^k} + \sqrt{n} = \sqrt{A^2 + (\sqrt{1982} - 1)^k(\sqrt{1982} + 1)^k + A} = \sqrt{A^2 + (B - A)(A + B) + A} = \sqrt{A^2 + B^2 - A^2 + A} = B + A = (\sqrt{1982} + 1)^k.
\]

**Case 2:** \( k \) is even. Let \( n = B^2 \). Then

\[
\sqrt{n + 1981^k} + \sqrt{n} = \sqrt{B^2 + (A - B)(A + B) + \sqrt{B^2}} = \sqrt{A^2 + \sqrt{B^2}} = (\sqrt{1982} + 1)^k.
\]

It is evident that \( A^2 \) is an integer, and \( B^2 \) is an integer since \( B \) has the form \( \sqrt{1982} L \), for some integer \( L \).

Editor’s note. Nicos Diamantis, student, University of Patras, Greece, also solved the problem. His method was to derive that a real solution is an integer.

10. In a given tetrahedron \( ABCD \), \( \angle BAC + \angle BAD = 180^\circ \). If \( AK \) is the bisector of \( \angle CAD \), determine \( \angle BAK \). (Grade 10)

Solution by Hans Engelhaupt, Gundelsheim, Germany.

If the triangle \( ABD \) is rotated with axis \( AB \), the point \( D \) describes a circle in a plane orthogonal to \( AB \). The bisector of \( \angle CAD \) meets \( CD \) at \( K \). Then \( K \) divides \( CD \) in the constant ratio \( AC/AD \). Thus \( K \) describes a circle in a plane parallel to the plane described by \( D \). Now \( A \) is a point of this circle [the condition \( \angle BAC + \angle BAD = 180^\circ \) means that at some stage in the rotation of \( \triangle ABD \) the point \( A \) will lie on the line \( CD \)]; therefore \( \angle BAK = 90^\circ \).
11. Show that it is possible to place non-zero numbers at the vertices of a given regular \( n \)-gon \( P \) so that for any set of vertices of \( P \) which are vertices of a regular \( k \)-gon \( (k \leq n) \), the sum of the corresponding numbers equals zero. (Grade 10)

**Solution by Nicos Diamantis, student, University of Patras, Greece.**

Consider a coordinate system with \( O(0,0) \) the centre of the \( n \)-gon (and therefore the centre of all regular \( k \)-gons \( (k \leq n) \) which have vertices some of those of the \( n \)-gon). At the vertices of the \( n \)-gon, place the \( x \)-coordinates. [The \( n \)-gon can be rotated so that none of these \( x \)-coordinates are zero.] But we have \( \sum_{i=1}^{k} \overrightarrow{OA}_i = \mathbf{0} \), where the \( A_i \) are the vertices of a regular \( k \)-gon. From this, looking at the \( x \)-coordinates we have \( \sum_{i=1}^{k} x_i = 0 \), where \( x_1, x_2, \ldots, x_k \) are the \( x \)-coordinates of \( A_1, \ldots, A_k \). This solves the problem.

*

Before turning to more recent problems, I want to apologize for leaving S.R. Cavior of the University of Buffalo off the list of solvers when I discussed problem 1 of the 24th Spanish Olympiad [1989: 67] in the January number [1991: 9]. His solution somehow had found its way into the collection for a later month.

*

For the remainder of this column, we turn to problems given in the June 1989 number of the Corner. We give solutions to all but numbers 3 and 6 of the 3rd Ibero-American Olympiad [1989: 163-164].

1. The angles of a triangle are in arithmetical progression. The altitudes of the triangle are also in arithmetical progression. Show that the triangle is equilateral.

**Solution by Bob Prielipp, University of Wisconsin-Oshkosh.**

Let \( A, B, C \) be the angles of the given triangle and let \( h_a, h_b, h_c \) be the corresponding altitudes. Without loss of generality, we may assume \( A \leq B \leq C \). Since the angles are in arithmetic progression \( A + C = 2B \), and since \( A + B + C = 180^\circ \), \( B = 60^\circ \). Now also \( h_c \leq h_b \leq h_a \) and \( a \leq b \leq c \) where \( a, b, c \) are the side lengths opposite \( A, B, C \), respectively.

From the law of cosines \( b^2 = a^2 + c^2 - 2ac \cos 60^\circ = a^2 + c^2 - ac \). Now \( 2b = h_a + h_c \) implies that \( 4F/b = 2F/a + 2F/c \), where \( F \) is the area of triangle \( ABC \), so

\[
\frac{2}{b} = \frac{1}{a} + \frac{1}{c} = \frac{a + c}{ac}, \quad \text{or} \quad b = \frac{2ac}{a + c}.
\]

Since \( b^2 = a^2 + c^2 - ac \), \( 4a^2c^2 = (a + c)^2(a^2 + c^2 - ac) \). From this we get

\[
0 = (a + c)^2(a^2 + c^2 - ac) - 4a^2c^2
= [(a - c)^2 + 4ac][(a - c)^2 + ac] - 4a^2c^2
= (a - c)^4 + 5ac(a - c)^2
= (a - c)^2[(a - c)^2 + 5ac].
\]

and so \( a = c \). This gives \( a = b = c \) since \( a \leq b \leq c \), and the given triangle is equilateral.

**Editor’s note.** The problem was solved using the law of sines by Michael Selby, University of Windsor.
2. Let \(a, b, c, d, p\) and \(q\) be natural numbers different from zero such that

\[
ad - bc = 1 \quad \text{and} \quad \frac{a}{b} > \frac{p}{q} > \frac{c}{d}.
\]

Show that

(i) \(q \geq b + d\);

(ii) if \(q = b + d\) then \(p = a + c\).

**Solution by Michael Selby, University of Windsor.**

Since \(a/b > p/q, aq - pd > 0\), so \(aq - pd \geq 1\). Likewise \(pd - cq \geq 1\). Now

\[
q = q(ad - bc) = b(pd - cq) + d(aq - pb) \geq b + d.
\]

If \(q = b + d\), then \(b(pd - cq - 1) + d(ap - bp - 1) = 0\). Since \(b > 0, d > 0, pd - cq - 1 \geq 0\) and \(aq - bp - 1 \geq 0\) we must have \(pd - cq = aq - pb = 1\). This yields \(q(a + c) = (d + b)p\).

But \(q = b + d\), so \(a + c = p\) as required.

4. Let \(ABC\) be a triangle with sides \(a, b, c\). Each side is divided in \(n\) equal parts. Let \(S\) be the sum of the squares of distances from each vertex to each one of the points of subdivision of the opposite side (excepting the vertices). Show that

\[
\frac{S}{a^2 + b^2 + c^2}
\]

is rational.

**Solution by Michael Selby, University of Windsor, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.**

Consider the triangle shown with sides represented by vectors. Let \(B_j\) be the vector joining the vertex \(B\) to the corresponding point of subdivision on side \(b\), and define \(C_j\) and \(A_j\) analogously. Then

\[
B_j = a + \frac{j}{n}b,
\]

so

\[
|B_j|^2 = |a|^2 + \frac{j^2}{n^2}|b|^2 + \frac{2j}{n}(a \cdot b).
\]

Therefore

\[
\sum_{j=1}^{n-1} |B_j|^2 = (n-1)a^2 + b^2 \sum_{j=1}^{n-1} \frac{j^2}{n^2} + \frac{2}{n} \sum_{j=1}^{n-1} j(a \cdot b)
\]

\[
= (n-1)a^2 + b^2 \frac{(n-1)(2n-1)}{6n} + (n-1)(a \cdot b).
\]

Similarly

\[
\sum_{j=1}^{n-1} |C_j|^2 = (n-1)b^2 + c^2 \frac{(n-1)(2n-1)}{6n} + (n-1)(b \cdot c),
\]
\[
\sum_{j=1}^{n-1} |A_j|^2 = (n-1)c^2 + a^2\frac{(n-1)(2n-1)}{6n} + (n-1)(a \cdot c).
\]

Therefore

\[
S = \sum_{j=1}^{n-1} (|A_j|^2 + |B_j|^2 + |C_j|^2) = (n-1)(a^2 + b^2 + c^2) + \frac{(a^2 + b^2 + c^2)(n-1)(2n-1)}{6n} + (n-1)(a \cdot b + b \cdot c + a \cdot c) \tag{1}
\]

Since \(a + b + c = 0\),

\[
0 = |a + b + c|^2 = a^2 + b^2 + c^2 + 2(a \cdot b + a \cdot c + b \cdot c).
\]

Substituting this into (1) gives

\[
S = (a^2 + b^2 + c^2) \left( n - 1 + \frac{(n-1)(2n-1)}{6n} - \frac{n-1}{2} \right).
\]

Hence

\[
\frac{S}{a^2 + b^2 + c^2} = \frac{(n-1)(5n-1)}{6n}.
\]

5. We consider expressions of the form

\[
x + yt + zt^2,
\]

where \(x, y, z \in \mathbb{Q}\), and \(t^2 = 2\). Show that, if \(x + yt + zt^2 \neq 0\), then there exist \(u, v, w \in \mathbb{Q}\) such that

\[
(x + yt + zt^2)(u + vt + wt^2) = 1.
\]

Solution by Michael Selby, University of Windsor.

Observe that \((x + yt + zt^2)(x + zt^2 - yt) = (x + 2z)^2 - 2y^2\). Now \((x + 2z)^2 - 2y^2 \neq 0\), for otherwise \(\sqrt{2} = |(x + 2z)/y|\) is rational. [Note if \(y = 0\) in this case, then \(x + 2z = 0\) and \(x + yt + zt^2 = 0\), contrary to assumption.] Let \(\alpha = (x + 2z)^2 - 2y^2\). Let \(u = x/\alpha\), \(v = -y/\alpha\) and \(w = z/\alpha\). Then

\[
(x + yt + zt^2)(u + vt + wt^2) = \frac{(x + 2y)^2 - 2y^2}{\alpha} = 1.
\]

Editor’s note. The result is still true if \(t^3 = 2\) replaces \(t^2 = 2\) in the problem.

* * * * *

Send me your nice solutions, and also your national and regional contests.

* * * * * * *
PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1991, although solutions received after that date will also be considered until the time when a solution is published.

Quadrilateral $ABCD$ is inscribed in circle $\omega$, with $AD < CD$. Diagonals $AC$ and $BD$ intersect in $E$, and $M$ lies on $EC$ so that $\angle CBM = \angle ACD$. Show that the circumcircle of $\triangle BME$ is tangent to $\omega$ at $B$.

1642. Proposed by Murray S. Klamkin, University of Alberta.
Determine the maximum value of
$$x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - x^2) + z(1 - x^2)(1 - y^2)$$
subject to $yz + zx + xy = 1$ and $x, y, z \geq 0$.

1643. Proposed by Toshio Seimiya, Kawasaki, Japan.
Characterize all triangles $ABC$ such that
$$\frac{AI_a}{BI_b} : \frac{CI_c}{BC} = \frac{CA}{\omega} : \frac{AB}{\omega},$$
where $I_a, I_b, I_c$ are the excenters of $\triangle ABC$ corresponding to $A, B, C$, respectively.

1644. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that it attains both positive and negative values, and let $n \geq 2$ be an integer. Show that there exists a strictly increasing arithmetic sequence $a_1 < \cdots < a_n$ such that $f(a_1) + \cdots + f(a_n) = 0$.

1645. Proposed by J. Chris Fisher, University of Regina.
Let $P_1, P_2, P_3$ be arbitrary points on the sides $A_2A_3, A_3A_1, A_1A_2$, respectively, of a triangle $A_1A_2A_3$. Let $B_1$ be the intersection of the perpendicular bisectors of $A_1P_2$ and $A_1P_3$, and analogously define $B_2$ and $B_3$. Prove that $\triangle B_1B_2B_3$ is similar to $\triangle A_1A_2A_3$.

Find all positive integers $n$ such that the polynomial
$$(a - b)^{2n}(a + b - c) + (b - c)^{2n}(b + c - a) + (c - a)^{2n}(c + a - b)$$
has $a^2 + b^2 + c^2 - ab - bc - ca$ as a factor.
1647. Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.

B and C are fixed points and A a variable point such that \( \angle BAC \) is a fixed value. 
D and E are the midpoints of AB and AC respectively, and F and G are such that 
\( FD \perp AB, GE \perp AC \), and FB and GC are perpendicular to BC. Show that \( |BF| \cdot |CG| \)

is independent of the location of A.


Evaluate \( \lim_{n \to \infty} (u_n/\sqrt{n}) \), where \( \{u_n\} \) is defined by \( u_0 = u_1 = u_2 = 1 \) and

\[ u_{n+3} = u_{n+2} + \frac{u_n}{2n + 6}, \quad n = 0, 1, \ldots. \]

1649*. Proposed by D.M. Milošević, Pranjani, Yugoslavia.

Prove or disprove that

\[ \sum \cot \frac{\alpha}{2} - 2 \sum \cot \alpha \geq \sqrt{3}, \]

where the sums are cyclic over the angles \( \alpha, \beta, \gamma \) of a triangle.

1650. Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.

Find all real numbers \( \alpha \) for which the equality

\[ [\sqrt{n + \alpha} + \sqrt{n}] = [\sqrt{4n + 1}] \]

holds for every positive integer \( n \). Here \([ \ ]\) denotes the greatest integer function. (This problem was inspired by problem 5 of the 1987 Canadian Mathematics Olympiad [1987: 214].)

* * * * *

**SOLUTIONS**

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


Can a square be dissected into three congruent nonrectangular pieces?

II. Solution by Sam Maltby, student, University of Calgary.

We show that the answer is no.

Let the unit square \( ABCD \) be cut into three congruent pieces \( P_1, P_2 \) and \( P_3 \), and suppose these pieces are not rectangles. Here we will assume that our “pieces” contain their boundaries, and therefore that two pieces are allowed to overlap on their boundaries but not elsewhere. We must also assume that the pieces are connected, and moreover that they contain no “isthmuses” or “tails”; otherwise it is easy to find counterexamples, as suggested by the following figures.
The effect of these assumptions is that we take the pieces to have the following three properties:

(1) If a piece contains a vertex of the square but does not contain at least part of both edges at that vertex, then some other piece must also contain that vertex;
(2) A segment of the boundary of the square cannot belong to more than one piece;
(3) Between any two points $X$ and $Y$ of a piece $P$ there is a curve (called an *incurve* in $P$ between $X$ and $Y$) with $X$ and $Y$ as endpoints such that all points of the curve other than $X$ and $Y$ are interior points of $P$. An incurve of one piece cannot intersect another piece except possibly at its endpoints.

Note that since the pieces are congruent it must be possible through some combination of translation, rotation or reflection to place one piece on another. This will take each curve in the first piece to some curve in the second; we shall say that these two curves *correspond*.

We now give a series of lemmas and eventually arrive at the desired conclusion.

**Lemma 1.** The intersection of a piece with a side of the square is connected.

*Proof.* Suppose for a contradiction that the intersection of one of the pieces, say $P_1$, with one of the sides of the square, say $AB$, is not connected. Then there are two points $E$ and $F$ on $AB$ in $P_1$ such that $EF$ is not entirely in $P_1$; say that point $G$ between $E$ and $F$ is in $P_2$. Let $c$ be an incurve in $P_1$ between $E$ and $F$. If there is some point $H$ in $P_2$ on the other side of $c$ from $G$, then an incurve in $P_2$ between $G$ and $H$ must intersect $c$, but it cannot have a common endpoint with $c$. This contradicts (3), so $H$ cannot exist. Thus $P_2$ is enclosed by $c$ and $EF$. But then the convex hull of $P_1$ is larger than the convex hull of $P_2$, which contradicts $P_1$ and $P_2$ being congruent. $\square$

**Lemma 2.** No piece contains two opposite corners of the square.

*Proof.* Suppose that, for instance, $P_1$ contains $A$ and $C$. Then clearly $P_2$ and $P_3$ must each also contain opposite corners of the square. If $P_2$ contains $B$ and $D$, then there is an incurve in $P_1$ between $A$ and $C$ and one in $P_2$ between $B$ and $D$. These must intersect, but they do not have a common endpoint, which contradicts (3). Thus $P_2$ and likewise $P_3$ must contain $A$ and $C$.

Now some piece, say $P_1$, contains $B$. Then $P_2$ and $P_3$ must each contain a point at distance 1 from both $A$ and $C$. But $B$ and $D$ are the only such points, so two of the
pieces (say $P_1$ and $P_2$) contain, say, $B$. By Lemma 1, both $P_1$ and $P_2$ contain $BC$, which is impossible by (2). □

By the Pigeonhole Principle, one of the three pieces, say $P_1$, contains at least two corners of the square, and by Lemma 1 these corners must be adjacent, say $A$ and $B$. By Lemma 1 $P_1$ contains the entire side $AB$. Then each of $P_2$ and $P_3$ contains a segment which corresponds to $AB$; call these segments $S_2$ and $S_3$ respectively. It follows that

(4) $P_2$ is contained in the area enclosed by $S_2$ and two rays with endpoints at the ends of $S_2$ and which are perpendicular to $S_2$; and likewise for $P_3$ and $S_3$.

**Lemma 3.** $S_2$ (and similarly $S_3$) cannot be parallel to $BC$.

**Proof.** Suppose $S_2$ is parallel to $BC$. If it is not $BC$ or $AD$ then there is an incurve in $P_2$ between the endpoints of $S_2$ which intersects one in $P_1$ between $A$ and $B$. This contradicts (3), so $S_2$ must be $BC$ or $AD$, say $BC$. Then since $AB$ and $BC$ correspond, $P_2$ must be either the reflection of $P_1$ through $BD$ or the rotation of $P_1$ through $90^\circ$ about the centre of the square.

In the former case, $P_3$ must be symmetric across $BD$, so $P_1$ must also be symmetric through some axis. Since the reflection cannot take $AB$ outside the square, the axis must be (i) $AC$, (ii) the perpendicular bisector of $AB$, (iii) a line parallel to $AB$, or (iv) a line through either $A$ or $B$ which makes an angle less than $45^\circ$ with $AB$.

If (i), then $P_1$ contains $B$ and $D$, which contradicts Lemma 2. If (ii), then no point on $AD$ other than $A$ can be in $P_1$; for by Lemma 1 some line segment in $AD$ would be in $P_1$, and thus (by reflection) a segment in $BC$ would be in $P_2$ as well as $P_1$, which is impossible by (2). Since $A$ cannot be in $P_2$ by Lemma 2, by (1) $A$ is in $P_3$. By the symmetry of $P_3$, $C$ must also be in $P_3$, which contradicts Lemma 2. If (iii), the reflection takes $B$ to some other point on $BC$, so $P_1$ contains some segment of $BC$ which is also in $P_2$, again a contradiction.

Finally, suppose (iv). If the axis is through $A$ then $P_1$ contains no point of $AD$ other than $A$, so $P_3$ contains $A$ and (by reflection) $C$, contrary to Lemma 2 as before. So the axis is through $B$; let $E$ be the reflection of $A$ through the axis. Since $P_1$ is bounded by $AD$, which is perpendicular to $AB$, it is also bounded by the perpendicular to $BE$ at $E$. Let $F$ be the intersection of the perpendicular and $AD$. Then $P_3$ contains $DF$ and by reflection a segment of equal length along $DC$. Now $\angle ABF \leq 22.5^\circ$, so $|AF| \leq \tan 22.5^\circ = \sqrt{2}-1$, so $|DF| = 1 - |AF| \geq 2 - \sqrt{2}$. Then $P_1$ must have two segments of length at least $2-\sqrt{2}$ in its boundary which are at right angles (corresponding to $DF$ and its reflection) and so that one is the image of the other across the axis of reflection $BF$. It is easy to see that this is impossible. Therefore $P_2$ is not a reflection of $P_1$.

If $P_2$ is a rotation of $P_1$ then $P_1$ cannot contain any line segment in $AD$, since by the rotation $P_2$ would contain some line segment in $AB$ which would also be in $P_1$, contrary
to (2). As before, \( P_2 \) cannot contain \( A \) and so \( A \) is in \( P_3 \) by (1). Likewise \( C \) is also in \( P_3 \), which again contradicts Lemma 2. \( \square \)

**Lemma 4.** \( P_2 \) cannot contain a side of the square unless that side is parallel to \( S_2 \). Likewise for \( P_3 \) and \( S_3 \).

**Proof.** First suppose that \( P_2 \) contains \( BC \). By Lemma 3 \( BC \) is not parallel to \( S_2 \), so by (4) \( S_2 \) must have either \( B \) or \( C \) as one of its endpoints. There are now four cases.

**Case (a).** \( B \) is an endpoint of \( S_2 \) and \( B \) in \( P_2 \) corresponds to \( A \) in \( P_1 \).

There is a segment \( AE \) in \( P_1 \) which corresponds to \( BC \) in \( P_2 \) such that \( \angle EAB \) is equal to the angle between \( S_2 \) and \( BC \). If \( AE \) and \( S_2 \) don’t intersect, then, extending them until they meet, we get a right triangle with hypotenuse \( AB \) of length 1 and one side of length at least 1, which is impossible. So they must intersect at some point interior to both, which is impossible by (3).

**Case (b).** \( C \) is an endpoint of \( S_2 \) and \( C \) in \( P_2 \) corresponds to \( A \) in \( P_1 \).

There must be a segment \( AE \) in \( P_1 \) which corresponds to \( BC \) in \( P_2 \). Then \( \angle EAB \) is equal to the angle made by \( S_2 \) and \( BC \); but then since \( S_2 \) is not \( BC \) or \( CD \) we see that \( S_2 \) and \( AE \) intersect at some point other than \( A \) or \( E \), which is impossible.

**Case (c).** \( C \) is an endpoint of \( S_2 \) and \( C \) in \( P_2 \) corresponds to \( B \) in \( P_1 \).

The argument here is essentially the same as that in case (a).

**Case (d).** \( B \) is an endpoint of \( S_2 \) and \( B \) in \( P_2 \) corresponds to \( B \) in \( P_1 \).

Let \( E \) be the other endpoint of \( S_2 \). Note that by Lemma 2 neither \( P_1 \) nor \( P_2 \) contains \( D \), so \( P_3 \) must, and so \( P_3 \) does not contain \( B \). Then \( S_2 \) must bisect the right angle \( ABC \), and \( P_2 \) must be a 45° rotation of \( P_1 \) around \( B \). Since \( P_2 \) is bounded by \( S_2 \), \( BC \), and the perpendicular to \( S_2 \) through \( E \), \( P_3 \) must contain the segment \( DF \) along \( DC \) with length \( 2 - \sqrt{2} \). Since \( P_2 \) is contained in \( BEFC \), by rotation \( P_1 \) is contained in \( ABEG \), so \( P_3 \) also contains the segment \( DG \) of length \( 2 - \sqrt{2} \), where \( EF \) meets \( AD \) at \( G \).

Now \( S_3 \) is not parallel to \( BC \) by Lemma 3, and it cannot be parallel to \( CD \) because if it were it would have to be \( CD \) itself, but then we could replace \( AB \) by \( CD \) and \( P_1 \) by \( P_3 \) in case (b) or (c) to get the result. So \( S_3 \) makes an acute angle with \( CD \). By applying (4) to \( P_3 \) we see that \( G, D \) and \( F \) must all be on the same side of \( S_3 \), but \( S_3 \) cannot intersect the interior of \( EB \) since \( EB \) lies on the boundary of \( P_1 \) and \( P_2 \). Thus \( S_3 \) must be \( GF \); but \( |GF| = 2(\sqrt{2} - 1) < 1 \), a contradiction. This finishes the proof that \( P_2 \) (and \( P_3 \)) cannot contain \( BC \). Similarly, \( P_2 \) and \( P_3 \) cannot contain \( AD \).

Now suppose that \( P_2 \) contains \( CD \) but \( S_2 \) is not parallel to \( CD \). By (4) \( S_2 \) must have either \( C \) or \( D \) as an endpoint, say \( C \). \( S_2 \) cannot be perpendicular to \( CD \) since it would then be \( BC \) contrary to Lemma 3. But since \( P_1 \) does not contain \( C \), \( P_3 \) contains \( C \) by (1). Now \( C \) in \( P_2 \) cannot correspond to \( A \) in \( P_1 \), since then \( P_1 \) would contain no point of \( AD \) other than \( A \), and \( A \) cannot be in \( P_2 \), so \( P_3 \) would contain \( A \) and \( C \) contrary to Lemma 2. So \( C \) in \( P_2 \) corresponds to \( B \) in \( P_1 \). But then \( P_3 \) contains \( B \) and thus \( BC \),
which is impossible by the above. □

**Lemma 5.** One of $P_2$ and $P_3$ must contain a side of the square.

*Proof.* To obtain a contradiction suppose that neither $P_2$ nor $P_3$ contains a side of the square. Then $S_2$ and $S_3$ are not sides of the square and so must be inside the square. First note that $P_1$ contains neither $C$ nor $D$ by Lemma 2, and neither $P_2$ nor $P_3$ contains both since it would then contain $CD$. Thus one, say $P_2$, contains $C$ and the other ($P_3$) contains $D$.

Also note that $P_1$ contains some point of $AD$ other than $A$ since otherwise $P_3$ would contain the entire side $AD$. Likewise $P_1$ contains some point of $BC$. By congruence then, there are right angles in $P_2$ ($P_3$) at both ends of $S_2$ ($S_3$).

Now $S_2$ is not parallel to $CD$ because if it were then $P_1$ would be a rectangle, contrary to assumption. Neither is it perpendicular, by Lemma 3. Thus by (4) $S_2$ must be oriented as shown. But then $C$ is the unique point in $P_3$ farthest from $S_2$. Thus there is one point of $P_1$ at maximum distance from $AB$ and so $P_1$ cannot contain more than one point of $CD$. Then $P_2$ and $P_3$ have a common point $E$ on $CD$, where we assume without loss of generality that $E$ is no farther from $D$ than from $C$. By the orientation of $S_2$ at least one end, call it $F$, of $S_2$ must be strictly closer to $AD$ than $E$ is. (Note $E \neq F$.)

Since (as above) $D$ is the only point in $P_3$ at maximum distance from $S_3$, $D$ in $P_3$ corresponds to $C$ in $P_2$. Thus $P_2$ is either a reflection of $P_3$ through the perpendicular bisector of $CD$ or a rotation of $P_3$ through $90^\circ$ about the centre of the square.

If it is a reflection then no interior point of $P_2$ or of $P_3$ is on the perpendicular bisector of $CD$ since it would then belong to both pieces. But there is an incurve from $F$ to $C$, and since $F$ is closer to $AD$ than $E$ is this curve must cross the bisector. This is impossible, so $P_2$ is not a reflection of $P_3$.

So $P_3$ must be a rotation of $P_3$ through $90^\circ$ about the centre of the square. Then $S_2$ is perpendicular to $S_3$, and since neither $S_2$ nor $S_3$ are sides of the square it follows that $S_2$ and $S_3$ must cross at a point interior to both. (Otherwise we can rotate both lines by $180^\circ$ and extend the resulting four lines until they meet, as in the diagram. Let $x$ be the length of the segment past the point of intersection. The convex hull of this figure is a square of side at least $\sqrt{1+x^2}$, and so has area at least $1+x^2$, but is contained in the unit square. Thus $x$ must be 0, which means that the original lines were sides of the square.) Since $S_2$ and $S_3$ are on the boundaries of $P_2$ and $P_3$ respectively, this is impossible. □

Now by Lemma 5, $P_2$ (say) contains a side of the square, but by Lemmas 3 and 4 it does not contain $BC$ or $AD$, so it must contain $CD$. By Lemma 4 $S_2$ is parallel to $CD$. If $CD$ is not $S_2$ then $P_2$ is a rectangle; therefore $CD$ is $S_2$ and then $P_1$ is either a reflection of $P_2$ through the perpendicular bisector of $BC$ or a rotation of $P_2$ through $180^\circ$ around the centre of the square.
In the case of a reflection, \( P_3 \) must be symmetric through the perpendicular bisector of \( BC \). However \( P_3 \) cannot contain \( BC \) or \( AD \); therefore \( P_1 \) contains some points in \( AD \) and \( BC \) other than \( A \) and \( B \), and so \( S_3 \) must have the perpendiculars from its endpoints at least partly in \( P_3 \). By (4) and the symmetry of \( P_3 \), either (i) \( S_3 \) is at \( 45^\circ \) to the perpendicular bisector of \( BC \) with an endpoint on it, or (ii) \( S_3 \) is parallel to \( AB \) (it cannot be perpendicular to \( AB \) by Lemma 3). However if (i) then since \( S_3 \) has length 1 it passes outside the square, and if (ii) then either \( P_1 \) or \( P_2 \) is a rectangle, which means we have a rectangular dissection.

If \( P_1 \) is a rotation of \( P_2 \), then rotating the entire square takes \( S_3 \) to another side of length one parallel to \( S_3 \) which is also in \( P_3 \). Then \( P_1 \) must have a side of length one parallel to \( AB \) and is easily seen to be rectangular, so the dissection is again rectangular.

The result has now been proved.

---


Let \( 0 < t \leq 1/2 \) be fixed. Show that

\[
\sum \cos tA \geq 2 + \sqrt{2} \cos (t + 1/4)\pi + \sum \sin tA,
\]

where the sums are cyclic over the angles \( A, B, C \) of a triangle. [This generalizes Murray Klamkin’s problem E3180 in the Amer. Math. Monthly (solution p. 771, October 1988).]


Our solution is essentially the same as O.P. Lossers in the October 1988 Monthly.

Let

\[
V(A, B, C, t) = \sum \cos tA - \sum \sin tA.
\]

Then we see that

\[
V(A, B, C, t) = \cos tA + \cos tB - (\sin tA + \sin tB) + \cos tC - \sin tC
\]

\[
= 2\cos \frac{t(A - B)}{2} \left( \cos \frac{t(A + B)}{2} - \sin \frac{t(A + B)}{2} \right) + \cos tC - \sin tC
\]

\[
= 2\cos \frac{t(A - B)}{2} \left( \cos \frac{t(\pi - C)}{2} - \sin \frac{t(\pi - C)}{2} \right) + \cos tC - \sin tC.
\]

Note that

\[
V(A, B, 0, t) = 2\sqrt{2} \cos \frac{t(A - B)}{2} \cos \left( \frac{t\pi}{2} + \frac{\pi}{4} \right) + 1
\]

\[
\geq 2\sqrt{2} \cos \frac{t\pi}{2} \cos \left( \frac{t\pi}{2} + \frac{\pi}{4} \right) + 1
\]

\[
= \sqrt{2} \cos (\pi t + \frac{\pi}{4}) + 2.
\]
For fixed $C \neq 0$, $V(A, B, C, t)$ is minimized only when $A = 0$ or $B = 0$, and maximized only when $A = B$, because

$$\left| \frac{t(A + B)}{2} \right| \leq \frac{\pi}{4} \quad \Rightarrow \quad \cos \frac{t(A + B)}{2} > \sin \frac{t(A + B)}{2}. $$

By symmetry, $V(A, B, C, t)$ is minimized on the boundary $A \cdot B \cdot C = 0$ [in fact when two of $A, B, C$ are 0], and maximized only at $A = B = C = \pi/3$. We conclude that

$$2 + \sqrt{2}\cos(\pi t + \frac{\pi}{4}) \leq V(A, B, C, t) \leq 3\sqrt{2}\cos\left(\frac{\pi t}{3} + \frac{\pi}{4}\right).$$

The left-hand inequality answers the problem.

II. Solution by Marcin E. Kuczma, Warszawa, Poland.

Slightly more generally, we assert that

$$\sum_{i=1}^{n} (\cos x_i - \sin x_i) \geq (n - 1) + \sqrt{2}\cos \left(\frac{\pi}{4} + \sum_{i=1}^{n} x_i\right) \quad (1)$$

for $x_1, \ldots, x_n \geq 0$, $\sum_{i=1}^{n} x_i \leq \pi/2$. For $n = 3$, $x_1 = tA, x_2 = tB, x_3 = tC$, this is the inequality of the problem.

Defining

$$f(x) = 1 + \sin x - \cos x = 1 - \sqrt{2}\cos\left(\frac{\pi}{4} + x\right),$$

we rewrite (1) as

$$f\left(\sum_{i=1}^{n} x_i\right) \geq \sum_{i=1}^{n} f(x_i), \quad (2)$$

for $x_1, \ldots, x_n \geq 0$, $\sum_{i=1}^{n} x_i \leq \pi/2$. (This is a yet more natural form of the statement.)

It suffices to prove (2) for $n = 2$; obvious induction then does the rest. And for $n = 2$, writing $x$ and $y$ for $x_1$ and $x_2$, we have

$$f(x + y) - f(x) - f(y) = (\cos y - \cos(x + y)) - (1 - \cos x) + (\sin(x + y) - \sin y) - \sin x$$

$$= \left(2\sin\left(y + \frac{x}{2}\right)\sin\frac{x}{2} - 2\sin^2\frac{x}{2}\right) + \left(2\cos\left(y + \frac{x}{2}\right)\sin\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$= 2\sin\frac{x}{2} \left(2\cos\frac{x + y}{2}\sin\frac{y}{2} - 2\sin\frac{x + y}{2}\sin\frac{y}{2}\right)$$

$$= 4\sin\frac{x}{2}\sin\frac{y}{2} \left(\cos\frac{x + y}{2} - \sin\frac{x + y}{2}\right) \geq 0$$

because $(x + y)/2 \in [0, \pi/4]$.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; and the proposer. There was one incorrect solution submitted.

The problem was also proposed independently (without solution) by Robert E. Shafer, Berkeley, California.

ABC is a triangle with sides a, b, c and area F, and P is an interior point. Put \( R_1 = AP, \) \( R_2 = BP, \) \( R_3 = CP. \) Prove that the triangle with sides \( aR_1, bR_2, cR_3 \) has circumradius at least \( 4F/(3\sqrt{3}). \)

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

In any triangle, \( R \geq \frac{1}{3\sqrt{3}}(a + b + c) \) ([1], item 5.3). Thus, using item 12.19 of [1], we get for the circumradius \( \tilde{R} \) under consideration:

\[
\tilde{R} \geq \frac{1}{3\sqrt{3}}(aR_1 + bR_2 + cR_3) \geq \frac{2}{3\sqrt{3}}(ar_1 + br_2 + cr_3)
\]

\[
= \frac{2}{3\sqrt{3}}(2F) = \frac{4F}{3\sqrt{3}}.
\]

Reference:

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer.

Klamkin and Kuczma note that inequality (1) is equivalent to the fact that the perimeter of a triangle inscribed in a circle is maximized when the triangle is equilateral.


Let \( m, n \) be given positive integers and \( d \) be their greatest common divisor. Let \( x = 2^m - 1, y = 2^n + 1. \)

(a) If \( m/d \) is odd, prove that \( x \) and \( y \) are coprime.
(b) Determine the greatest common divisor of \( x \) and \( y \) when \( m/d \) is even.

Solution by Kenneth M. Wilke, Topeka, Kansas.

Putting \( m_1 = m/d \) and \( n_1 = n/d, \) we have \((m_1, n_1) = 1. \) Let \( \delta = (x, y) \) and put \( k = x/\delta, \ell = y/\delta. \) We want to find \( \delta. \)

(a) If \( m_1 = m/d \) is odd, then

\[
(k\delta + 1)^{\ell} = 2^{m_1} = 2^{m_1 \ell d} = 2^{m_1} = (\ell\delta - 1)^{m_1}.
\]

But by the binomial theorem, \((k\delta+1)^{\ell} = K\delta + 1 \) for some integer \( K, \) and \((\ell\delta - 1)^{m_1} = L\delta - 1 \) for some integer \( L \) since \( m_1 \) is odd. Hence \( K\delta + 1 = L\delta - 1, \) or \( 2 = \delta(L - K). \) Thus \( \delta \mid 2 \) and since both \( x \) and \( y \) are odd, \( \delta \) must be odd also. Hence \( \delta = 1, \) as required.
(b) If $m_1 = m/d$ is even, say $m_1 = 2m_2$, then $(m_2, n_1) = 1$. We shall use the known result that for natural numbers $a, m, n$ such that $a > 1$, $(a^n - 1, a^n - a^{(m_n)}) = a^{(m_n)} - 1$. Let $y' = 2^n - 1$. Then $\delta|yy' = 2^{2n} - 1$, so $\delta|(x, yy')$. But

$$
(x, yy') = (2^{2m_2d} - 1, 2^{2m_1d} - 1) = 2^{(2m_2d, 2m_1d)} - 1
$$

$$
= 2^{2d} - 1 = (2^d + 1)(2^d - 1).
$$

Since $(m_1, n_1) = 1$ and $m_1$ is even, $n_1$ must be odd. Hence $(2^d + 1)|(2^{2n_1d} + 1) = y$. Also, if $r > 1$ is any divisor of $2^d - 1$, we have

$$
\frac{y}{r} = \frac{2^{n_1d} + 1}{r} = \frac{2^{n_1d} - 1}{r} + \frac{2}{r},
$$

where $r|(2^{n_1d} - 1)$, and hence $y/r$ is not an integer. Thus we must have $\delta = (x, y) = 2^d + 1$ in this case.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and the proposer.


Let $n$ and $q$ denote positive integers. The identity

$$
\sum_{k=1}^{n} k \binom{n}{k} q^{n-k} = n(q+1)^{n-1}
$$

can be proved easily from the Binomial Theorem. Establish this identity by a combinatorial argument.

Solution by H.L. Abbott, University of Alberta.

The Mathematics Department at a certain university has $n$ members. The administration of the department is handled by an executive committee whose chairman also serves as chairman of the department. There are no restrictions, except the obvious ones, on the size of the executive committee. For example, during those years when many onerous problems are expected to arise it may be a committee of one, while in times when little of any consequence needs attention it may consist of the whole department. Each member of the department who is not a member of the executive committee is required to serve on exactly one of $q$ committees. There is no restriction on the size of these committees either. Indeed, some of them need not have any members at all. This, for example, will be the case when the size $k$ of the executive committee is such that $q + k > n$. 

Late one evening as the chairman was leaving the department he remarked to his secretary that there is a simple expression for the number of possible administrative structures for the department. “Observe,” he said, “that the number of ways of choosing an executive committee of size \( k \) is \( \binom{n}{k} \). The chairman of this committee, and thus of the department, may then be chosen in \( k \) ways and the remaining \( n - k \) members of the department may then be assigned their tasks in \( q^{n-k} \) ways. Thus the number of possible bureaucracies is \( \sum_{k=1}^{n} k \binom{n}{k} q^{n-k} \).”

His secretary, almost without hesitation, replied, “Surely there is a much simpler expression for this number. The chairman of the executive committee may be chosen in \( n \) ways and after this choice has been made the remaining \( n - 1 \) members of the department may be assigned their administrative chores in \( (q + 1)^{n-1} \) ways. Thus the number is \( n(q + 1)^{n-1} \).”

A few moments later the chairman related this conversation to the caretaker as they rode the elevator to the first floor. “It shows,” said the caretaker, “that your secretary is the one who counts.”

A student who was in the elevator was overjoyed upon hearing this discussion. She realized that she could now solve the one remaining question on her combinatorics assignment which was due at the next class. The problem called for a combinatorial proof that

\[
\sum_{k=2}^{n} k(k-1) \binom{n}{k} q^{n-k} = n(n-1)(q + 1)^{n-2}.
\]

As soon as she arrived home she wrote out the solution: The Mathematics Department at a certain university has \( n \) members. The administration of the department is handled by an executive committee whose chairman and associate chairman also serve as chairman and associate chairman of the department....

Also solved by JACQUES CHONÉ, Clermont-Ferrand, France; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; TOM LEINSTER, Lancing College, England; and the proposers.

Also solved by JACQUES CHONÉ, Clermont-Ferrand, France; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; TOM LEINSTER, Lancing College, England; and the proposers.

\[ 1527. \text{[1990: 75]} \text{ Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.} \]

In quadrilateral \( ABCD \) the midpoints of \( AB, BC, CD \) and \( DA \) are \( P, Q, R \) and \( S \) respectively. \( T \) is the intersection point of \( AC \) and \( BD \), \( M \) that of \( PR \) and \( QS \), \( G \) is the centre of gravity of \( ABCD \). Show that \( T, M \) and \( G \) are collinear, and that \( TM: MG = 3:1 \).

Solution by John Rausen, New York.

The “center of gravity” of a quadrilateral is ambiguous. For a triangle, the centroid \( G \) (point of intersection of the medians) is a “center of gravity” in two different ways: (1) \( G \) is the center of mass of a system of three equal point-masses at the vertices; (2) \( G \) is also the center of mass of a uniform mass distribution on a thin plate (“lamina”) covering the triangle. In the case of a (plane) quadrilateral, we get two different points: (1) the center of mass of a set of four equal masses, say unit masses, at the vertices. This is point
$M$ of the statement because, by an elementary principle of mechanics, we can replace the unit masses at $A$ and $B$ by a mass of 2 units at the midpoint $P$ of $AB$, and similarly replace the unit masses at $C$, $D$ by a mass of 2 at point $R$; then the center of mass of the system is the midpoint of $PR$. But it is also the midpoint of $QS$, hence it is point $M = PR \cap QS$. Note that, by the same reasoning, $M$ is also the midpoint of the line segment connecting the midpoints $U, V$ of the diagonals $AC, BD$. $M$ is often called the centroid of the quadrilateral $ABCD$ [1].

Therefore the point $G$ of the problem must be (2) the center of mass of a uniform distribution over the surface of the quadrilateral (and the problem provides an interesting relation between the two "centers of gravity"). Point $G$ can be located by the same mechanical principle. Assuming first that $ABCD$ is convex, suppose the mass of triangle $BCD$ is concentrated at its centroid $A'$, and the mass of triangle $BAD$ at its centroid $C'$. Then, since the quadrilateral is the union of these two triangles (with no overlap), the center of mass $G$ is some point on line $A'C'$ (the exact position determined by the ratio of the areas of the two triangles). But by the same reasoning, $G$ is on line $B'D'$, where $B', D'$ are the centroids, respectively, of triangles $ADC, ABC$. Therefore point $G$ is the intersection of lines $A'C'$ and $B'D'$.

If the quadrilateral is not convex, in one case instead of the quadrilateral being the union of two triangles, we would have one of the triangles the union of the quadrilateral and the other triangle, but then the three centers of mass are still collinear, so the conclusion $G = A'C' \cap B'D'$ holds in all cases.

Returning to point $M$, it can also be obtained by first combining the unit masses at points $B, C, D$ into a mass of 3 units at the centroid $A'$ of triangle $BCD$. Then $M$ is the point on line $AA'$ such that $\overline{AM} = 3 \overline{MA'}$, or $\overline{MA'} = -\frac{1}{3} \overline{MA}$. Similarly, $\overline{MB'} = -\frac{1}{3} \overline{MB}$, $\overline{MC'} = -\frac{1}{3} \overline{MC}$ and $\overline{MD'} = -\frac{1}{3} \overline{MD}$. Therefore quadrilateral $A'B'C'D'$ is the image of quadrilateral $ABCD$ under the homothetic transformation with center $M$ and ratio $-1/3$, i.e., the point transformation which takes any point $P$ in the plane into point $P'$ defined by $\overline{MP'} = -\frac{1}{3} \overline{MP}$. Under such a transformation, any point associated to $ABCD$ goes into the corresponding point associated to $A'B'C'D'$. Thus, since $M$ goes into itself, it is also the centroid of $A'B'C'D'$. As to point $T = AC \cap BD$, its image is the point $A'C' \cap B'D' = G$, and it is true that $\overline{MG} = -\frac{1}{3} \overline{MT}$, or $\overline{TM} = 3 \overline{MG}$, as required.

Reference:

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALther JANOUS, Ursulengymnasium, Innsbruck, Austria; MARCin E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; TOM LEINSTER, Lancing College, England; P. PENNING, Delft, The Netherlands; Toshio SEIMIYA, Kawasaki, Japan; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

If \(a, b, c, d\) are positive real numbers such that \(a + b + c + d = 2\), prove or disprove that
\[
\frac{a^2}{(a^2 + 1)^2} + \frac{b^2}{(b^2 + 1)^2} + \frac{c^2}{(c^2 + 1)^2} + \frac{d^2}{(d^2 + 1)^2} \leq \frac{16}{25}.
\]

**Solution by G.P. Henderson, Campbellcroft, Ontario.**

We will prove that the inequality is true.

Set
\[
f(x) = \frac{x^2}{(x^2 + 1)^2}.
\]

Then
\[
f'(x) = \frac{2x(1 - x^2)}{(x^2 + 1)^3}, \quad f''(x) = \frac{2(3x^4 - 8x^2 + 1)}{(x^2 + 1)^4}.
\]

\(f\) has a minimum at \(x = 0\), a maximum at \(x = 1\), then decreases and approaches zero as \(x \to \infty\). There is a point of inflection at \(x = \sqrt{(4 - \sqrt{13})/3} \approx 0.36\).

The tangent at \(x = 1/2\) is
\[
y = \frac{4}{25} + \frac{48}{125} \left(x - \frac{1}{2}\right).
\]

Since \(f''(1/2) < 0\), the curve is below the tangent near \(x = 1/2\). At \(x = 0\), it is above the tangent. They intersect at \(x_1\), the real root of
\[
12x^3 + 11x^2 + 32x - 4 = 0.
\]

The polynomial is negative at \(x = 0\) and positive at \(x = 1/8\). Therefore \(x_1 < 1/8\).

Set
\[
F = f(a) + f(b) + f(c) + f(d), \quad 0 \leq a, b, c, d, \quad \sum a = 2.
\]

If \(a, b, c, d \geq x_1\),
\[
F \leq \sum \left[\frac{4}{25} + \frac{48}{125} \left(a - \frac{1}{2}\right)\right] = \frac{1}{125} \sum (48a - 4) = \frac{48}{125} \cdot 2 - \frac{4}{125} \cdot 4 = \frac{16}{25},
\]
as claimed.

Suppose now, that at least one of \(a, b, c, d < x_1\), say \(a < x_1\). Set \(t = (2 - a)/3\).

Since \(0 \leq a < 1/8, \; 5/8 < t < 2/3\).

For \(a\) fixed, we will show that the maximum \(F\) occurs at \(b = c = d = t\). The tangent at \(x = t\) has the form
\[
y = f(t) + m(x - t) \quad \text{(1)}
\]
At $x = t$, the curve is below the tangent because $t$ is greater than the abscissa of the point of inflection. It is easily verified that the tangent at $x = 1/\sqrt{3}$ passes through the origin. Since $t > 1/\sqrt{3}$, (1) is still above the curve at $x = 0$. Therefore

$$f(x) \leq f(t) + m(x - t).$$

Using this for $x = b, c, d$,

$$F \leq f(a) + 3f(t) + m(b + c + d - 3t) = f(a) + 3f(t).$$

It remains to show that for $t = (2 - a)/3$,

$$G(a) = f(a) + 3f(t) < 16/25.$$  

We have

$$G'(a) = f'(a) - f'(t) \leq \max_{0 \leq a \leq 3} f'(a) - \min_{\frac{2}{3} \leq t \leq 1} f'(t)$$

$$= f'(1/8) - f'(2/3) = 63 \cdot 64 \cdot 16 - \frac{20 \cdot 27}{13^3} < 0.$$  

Therefore

$$G(a) \leq G(0) = \frac{108}{169} < \frac{16}{25}.$$  

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; and JOHN LINDSEY, Northern Illinois University, Dekalb.

Equality of course holds for $a = b = c = d$. Engelhaupt notes that (with $f(x)$ defined as in Henderson's solution) it is not always true, for positive $a, b, c, d$ satisfying $a + b + c + d = k$, that

$$f(a) + f(b) + f(c) + f(d) \leq 4f(k/4).$$

For example, when $k = 1$, he obtains

$$f(0.2) + f(0.2) + f(0.3) + f(0.3) > 4f(0.25),$$

and when $k = 8$, he obtains

$$f(1.8) + f(1.8) + f(2.2) + f(2.2) > 4f(2).$$

He asks for which values of $k$ (2) is true.

* * * *
Let \( I_k = \frac{\int_0^{\pi/2} \sin^{2k} x \, dx}{\int_0^{\pi/2} \sin^{2k+1} x \, dx} \)
where \( k \) is a natural number. Prove that
\[
1 \leq I_k \leq 1 + \frac{1}{2k}
\]

Solution by Marcin E. Kuczma, Warszawa, Poland.

This is a variation on a theme of Wallis. The infinite product formula
\[
1 = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2k+1}{2k} \cdot \ldots
\]
is a lesson we’ve all been taught in the course of elementary calculus. The usual way it’s derived in textbooks is by considering the integrals
\[
c_n = \int_0^{\pi/2} \sin^n x \, dx
\]
and their basic recursion (resulting from integration by parts)
\[
c_n = \frac{n-1}{n} c_{n-2} \quad (c_0 = \pi/2, \ c_1 = 1).
\]

For \( I_k \) this yields the recursion
\[
I_k = \frac{c_{2k}}{c_{2k+1}} = \left( \frac{2k-1}{2k} \cdot c_{2k-2} \right) \left( \frac{2k}{2k+1} \cdot c_{2k-1} \right)^{-1} = \frac{2k-1}{2k} \cdot \frac{2k+1}{2k} I_{k-1}
\]
(with \( I_0 = \pi/2 \)), and hence we get for \( k = 1, 2, 3, \ldots \)
\[
I_k = \frac{\pi}{2} \left( \frac{1}{2} \cdot \frac{3}{2} \right) \left( \frac{3}{4} \cdot \frac{5}{4} \right) \cdots \left( \frac{2k-1}{2k} \cdot \frac{2k+1}{2k} \right).
\]
This is just a piece of (1). If we however truncate (1) one step earlier, we obtain another partial product of (1), which we denote \( J_k \). Thus
\[
J_k = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{2k-1}{2k}, \quad J_{k+1} = J_k \cdot \frac{2k+1}{2k} \cdot \frac{2k+1}{2k+2}.
\]

Relations (2) and (4) show that \( < I_k > \) is a decreasing sequence, \( < J_k > \) is increasing, and both converge to 1, in agreement with (1). So we have for each \( k \)
\[
I_k > 1 > J_k = \frac{2k}{2k+1} I_k.
\]
and this is exactly what we had to show.

With a little further effort we can obtain a much more precise two-sided estimate for $I_k$. By (1) and (3),

$$I_k = \prod_{j=k+1}^{\infty} \left( \frac{2j}{2j-1}, \frac{2j}{2j+1} \right).$$

(5)

It follows from the Lagrange (intermediate value) theorem that

$$\frac{1}{x} < \ln x - \ln(x-1) < \frac{1}{x-1} \quad \text{for } x > 1,$$

which with $x = 4j^2$ gives

$$\frac{1}{4j^2} < \ln \frac{4j^2}{4j^2-1} < \frac{1}{4j^2-1}.$$

Thus, by (5),

$$\ln I_k = \sum_{j=k+1}^{\infty} \ln \left( \frac{4j^2}{4j^2-1} \right) \left\{ \begin{array}{l}
\frac{1}{4j^2-1} = \frac{1}{4j^2-1} = \frac{1}{5} \\
\frac{1}{4j^2} = \frac{1}{4j^2} = \frac{1}{5} \\
\frac{1}{4j^2-1} = \frac{1}{4j^2-1} = \frac{1}{5} \\
\frac{1}{4j^2} = \frac{1}{4j^2} = \frac{1}{5} \\
\end{array} \right\}$$

and finally

$$I_k \left\{ \begin{array}{l}
< \exp \frac{1}{4k+2} = \exp \left( \frac{1}{4k+2} \right) < \frac{1}{4k+1}, \\
> \exp \frac{5}{20k+12} > 1 + \frac{5}{20k+12} + \frac{1}{2} \left( \frac{5}{20k+12} \right)^2 > 1 + \frac{1}{4k+2}.
\end{array} \right\}$$

(6)

Thus e.g. $1.0098 < I_{25} < 1.01$.

Note. Equality (3) can be rewritten as $I_k = \pi(k + \frac{1}{2})(2k)!/(2^k k!)$; consequently, the estimates (6) can be also derived by brute force (with much more calculation, though) from the Stirling formula, taken for instance in the form

$$n! = \sqrt{2\pi} n^n e^{-n} n^n , \quad \frac{12n+1}{12n} < \alpha_n < \frac{12n}{12n-1} .$$

Also solved by H.L. ABBOTT, University of Alberta; M. FALKOWITZ, Hamilton, Ontario; C. FESTRAETS-HAMOIR, Brussels, Belgium; GEORGE P. HENDERSON, Camperdown, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; TOM LEINSTER, Lancing College, England; JOHN LINDSEY, Northern Illinois University, Dekalb; BEATRIZ MARGOLIS, Paris, France; VEDULA N. MURTY, Perm State University at Harrisburg; P. PENNING, Delft, The Netherlands; COS PISCHETTOLA, Framingham, Massachusetts; ROBERT E. SHAFER, Berkeley, California; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Breda, The Netherlands; and KENNETH S. WILLIAMS, Carleton University.

Many of the solvers gave sharper bounds than the problem asked for. Kuczma's appear to be about the best with not a great deal of calculation. Henderson notes that $k$ need not be an integer.
The problem is an old one and, as Kuczma and others point out, comes from the usual proof of the Wallis product. Falkowitz found the given inequality, with solution, in R. Courant, Differential and Integral Calculus Vol I, pp. 223-224. Leinster spotted it in Spivak’s Calculus, page 328, problem 26.


Prove that

\[
\frac{v + w}{u} \cdot \frac{bc}{s - a} + \frac{w + u}{v} \cdot \frac{ca}{s - b} + \frac{u + v}{w} \cdot \frac{ab}{s - c} \geq 4(a + b + c),
\]

where \(a, b, c, s\) are the sides and semiperimeter of a triangle, and \(u, v, w\) are positive real numbers. (Compare with Crux 1212 [1988: 115].)


By the A.M.-G.M. inequality the left hand side of the given inequality is greater than or equal to

\[
3\sqrt[3]{\prod \left( \frac{u + v}{w} \right) \cdot \frac{a^2 b^2 c^2}{(s - a)(s - b)(s - c)}} = 3\sqrt[3]{\prod \left( \frac{u + v}{w} \right) \cdot \frac{16R^2r^2s^2}{r^2s}} = 3\sqrt[3]{\prod \left( \frac{u + v}{w} \right) \cdot 16R^2s},
\]

where \(R\) is the circumradius and \(r\) the inradius, and the products are cyclic over \(u, v, w\). So if I can show that

\[
3\sqrt[3]{\prod \left( \frac{u + v}{w} \right) \cdot 16R^2s} \geq 8s,
\]

the problem is solved. Cubing both sides gives

\[
\prod \left( \frac{u + v}{w} \right) \geq \frac{32}{27} \left( \frac{s}{R} \right)^2 = \frac{32}{27} \left( \sum \sin A \right)^2,
\]

where the sum is cyclic over the angles \(A, B, C\) of the triangle. Now it is known that

\[
\sum \sin A \leq \frac{3\sqrt{3}}{2},
\]

and also

\[
\prod \left( \frac{u + v}{w} \right) = \left( \frac{u}{w} + \frac{v}{w} \right) \left( \frac{v}{u} + \frac{w}{u} \right) \left( \frac{w}{v} + \frac{u}{v} \right).
\]

\[
= 2 + \left( \frac{u}{w} + \frac{w}{u} \right) + \left( \frac{v}{u} + \frac{u}{v} \right) + \left( \frac{w}{v} + \frac{v}{w} \right)
\]

\[
\geq 2 + 3 \cdot 2 = 8.
\]
Therefore
\[
\frac{32}{27} \left( \sum \sin A \right)^2 \leq \frac{32}{27} \left( \frac{3\sqrt{3}}{2} \right)^2 = 8 \leq \prod \left( \frac{u + v}{w} \right),
\]
which is (1). Equality holds when \(a = b = c\) and \(u = v = w\).

II. Generalization by Murray S. Klamkin, University of Alberta.

First we prove that
\[
\sum \frac{v + w}{u}(bc)^p \geq 6 \left( \frac{4F}{\sqrt{3}} \right)^{2p},
\]
where \(F\) is the area of the triangle, the sums here and subsequently are cyclic over \(u, v, w\) and \(a, b, c\), and for now \(p \geq 1\). Regrouping the left side of (2) and applying the A.M.-G.M. inequality to each resulting summand, we get
\[
\sum \frac{v + w}{u}(bc)^p = \sum \left( \frac{v}{u}(bc)^p + \frac{u}{v}(ca)^p \right) \geq 2(abc)^p(a^p + b^p + c^p).
\]

We now use \(abc = 4RF\) (\(R\) the circumradius) and the following known inequalities:
\[
a^p + b^p + c^p \geq \frac{(a + b + c)^p}{3^{p-1}}, \quad p \geq 1 \quad \text{(by the power mean inequality)};
\]
\[
R^2 \geq \frac{4F}{3\sqrt{3}} \quad \text{(the largest triangle inscribed in a circle is equilateral)};
\]
\[
(a + b + c)^2 \geq 12F\sqrt{3} \quad \text{(the largest triangle with given perimeter is equilateral)}.
\]

Stringing these together, we get
\[
\sum \left( \frac{v}{u} + \frac{w}{u} \right)(bc)^p \geq \frac{2(4RF)^p(a + b + c)^p}{3^{p-1}}
\]
\[
\geq 6(\frac{4F}{3})^p \left( \frac{4F}{3\sqrt{3}} \right)^{p/2} = 6 \left( \frac{4F}{\sqrt{3}} \right)^{2p},
\]
i.e., (2).

We now extend the range of (2) by showing that it is also valid for \(0 \leq p < 1\). The rest of the proof is similar to Janous’s solution of \(Crux\) 1212 [1988: 115-116] and uses results mentioned there. For \(0 \leq p < 1\), \(a^p, b^p, c^p\) are the sides of a triangle of area \(F_p \geq F^p(\sqrt{3}/4)^{1-p}\). From this and the case \(p = 1\) of (2), we get
\[
\sum \frac{v + w}{u}(bc)^p \geq 32F^2 \geq 6 \left( \frac{4F}{\sqrt{3}} \right)^{2p}.
\]

Now if \(a, b, c\) are the sides of a triangle, then so are
\[
\sqrt{a(s - a)}, \quad \sqrt{b(s - b)}, \quad \sqrt{c(s - c)},
\]
and the area of this triangle is \( F / 2 \). Hence from (2),

\[
\sum \frac{v + w}{u} \binom{bc(s - b)(s - c)}{s}^p \geq 6 \left( \frac{2F}{\sqrt{3}} \right)^{2p} \cdot
\]

Dividing by \( F^{2p} = (s(s - a)(s - b)(s - c))^p \), we obtain

\[
\sum \frac{v + w}{u} \left( \frac{bc}{s - a} \right)^p \geq 6 \left( \frac{4s}{3} \right)^p .
\]

The proposed inequality corresponds to the special case \( p = 1 \).

As a companion inequality, we obtain

\[
\sum \frac{v + w}{u} a^{2p} \geq 6 \left( \frac{4F}{\sqrt{3}} \right)^p
\]

for \( p \geq 0 \). We get as before (via regrouping and the A.M.-G.M. inequality) that

\[
\sum \frac{v + w}{u} a^{2p} = \sum \left( \frac{v}{u} a^{2p} + \frac{u}{v} b^{2p} \right) \geq 2 \sum (ab)^p .
\]

For \( p \geq 1 \), the rest follows from the known inequalities

\[
\sum (ab)^p \geq 3 \left( \frac{3ab}{3} \right)^p , \quad p \geq 1 \quad \text{(power mean)}
\]

and

\[
\sum ab \geq 4F\sqrt{3} .
\]

The extension of (4) to the range \( 0 \leq p < 1 \) is carried out the same way as (2) was extended.

By letting \( a = \sqrt{a(s - a)} \), etc. as before, we obtain a dual inequality to (4), i.e.,

\[
\sum \frac{v + w}{u} a^p (s - a)^p \geq 6 \left( \frac{2F}{\sqrt{3}} \right)^p .
\]

Finally, since we always have the representation

\[
a = y + z , \quad b = z + x , \quad c = x + y , \quad s = x + y + z , \quad F^2 = xyz(x + y + z) ,
\]

(2)—(5) take the forms

\[
\sum \frac{v + w}{u} (x + y)^{2p}(x + z)^{2p} \geq 6 \left( \frac{16xyz(x + y + z)}{3} \right)^p ,
\]

\[
\sum \frac{v + w}{u} \frac{(x + y)^{p}(x + z)^{p}}{x^p} \geq 6 \left( \frac{4(x + y + z)}{3} \right)^p ,
\]

\[
\sum \frac{v + w}{u} (y + z)^{4p} \geq 6 \left( \frac{16xyz(x + y + z)}{3} \right)^p ,
\]

\[
\sum \frac{v + w}{u} (y + z)^{2p} x^{2p} \geq 6 \left( \frac{4xyz(x + y + z)}{3} \right)^p ,
\]
respectively, for arbitrary nonnegative numbers \( x, y, z, u, v, w \) (we have doubled \( p \) in the last two inequalities). Numerous special cases can now be obtained.

III. Generalization by Walther Janous, Ursulengymnasium, Innsbruck, Austria. We start by proving the following Lemma. Let \( r_1, \ldots, r_n > 0 \) and put \( R = r_1 + \cdots + r_n \). Then for all nonnegative \( x_1, \ldots, x_n \),

\[
\sum_{i=1}^{n} \frac{R - r_i}{r_i} x_i^2 \geq 2 \sum_{i<j} x_i x_j .
\]

Proof. Indeed,

\[
\sum_{i=1}^{n} R - r_i R \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i^2 \geq \left( \sum_{i=1}^{n} x_i \right)^2 - \sum_{i=1}^{n} x_i^2 = 2 \sum_{i<j} x_i x_j ,
\]

where we have used the Cauchy-Schwarz inequality. Equality holds if and only if \( x_1/r_1 = \ldots = x_n/r_n \). □

Putting \( n = 3 \) and

\[
r_1 = u, \quad r_2 = v, \quad r_3 = w, \quad x_1 = bc, \quad x_2 = ca, \quad x_3 = ab
\]

we get

\[
\sum_{i=1}^{3} \frac{v + w}{u} (bc)^2 \geq 2 \sum caab = 2abc \sum a = 4abcs , \tag{6}
\]

where the sums are cyclic.

[Editor’s note. Janous now uses

\[
4abcs = 16FRs \geq 32Frs = 32F^2
\]

to obtain (2) for the case \( p = 1 \), then mimics his proof of Crux 1212 (exactly as Klamkin does) to obtain inequalities (2) and (3) for \( 0 \leq p \leq 1 \). Then he uses his lemma with \( n = 3, r_1 = u, r_2 = v, r_3 = w, x_1 = a^2, x_2 = b^2, x_3 = c^2, \) and

\[
x^2 + y^2 + z^2 \geq xy + yz + zx
\]

(which is equivalent to \((x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0\) to obtain

\[
\sum_{i=1}^{3} \frac{v + w}{u} a^4 \geq 2 \sum b^2 c^2 \geq 2 \sum caab = 4abcs \geq 32F^2 , \tag{7}
\]

and also extends this to obtain inequality (4) for \( 0 \leq p \leq 2 \) only. He points out that (4) can be considered a “dual” of the published generalization of Crux 1051 (inequality (2) on [1986: 252], due independently to him and Klamkin), in the same way that (3) is a “dual” of his generalization of Crux 1212 (see [1988: 116]) and (2) is a “dual” of Crux 1221 (see (3) on [1988: 116]). Janous now continues... ]
Obviously all the special cases stated in the solution of *Crux* 1051 can be translated literally into their “dual” versions, e.g., (v) on [1986: 254] becomes

\[
\sum \frac{2s + a}{2s - a} a^4 \geq 32F^2.
\]

Using the stronger inequality in (7), putting \( u = a^3 \), etc. and dividing by \( 2F = ah_a = bh_b = ch_c \), we get

\[
\sum \frac{b^3 + c^3}{h_a} \geq \frac{2abc}{F} = 8Rs.
\]

Etc., etc., etc.

Applying the transformation \( a \rightarrow \sqrt{a(s-a)} \), etc. to (6), we get

\[
\sum \frac{v + w}{u} bc(s-b)(s-c) \geq 2\sqrt{\prod a(s-a)} \sum \sqrt{a(s-a)},
\]

i.e.,

\[
\sum \frac{v + w}{a} bc(s-b)(s-c) \geq 2\sqrt{\frac{abc}{\prod(s-a)}} \sum \sqrt{a(s-a)} = 4\sqrt{\frac{R}{r}} \sum \sqrt{a(s-a)}.
\]

It seems that the right-hand quantity is greater than or equal to 8s (which if true would strengthen the proposed problem), but I can’t prove or disprove this. Therefore I leave to the readers the following

**Problem. Prove or disprove that**

\[
\sum \sqrt{\frac{a}{r_a}} \geq 2\sqrt{\frac{s}{R}}, \tag{8}
\]

where \( r_a, r_b, r_c \) are the exradii of the triangle.

Since \( s - a = rs/r_a \), etc., this inequality is equivalent to the one I can’t prove. Furthermore, (8) should be compared to item 5.47, p. 59 of Bottema et al, *Geometric Inequalities*, namely

\[
\sum \sqrt{\frac{a}{r_a}} \leq \frac{3}{2}\sqrt{\frac{s}{r}}.
\]

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; STEPHEN D. HNIDEI, student, University of British Columbia; MARCIN E. KUCZMA, Warszawa, Poland; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; and the proposer.

Milošević also proved inequality (7), with right-hand side 4abcs.
Crux Mathematicorum

Volume 17, Number 6       June 1991

CONTENTS

The Olympiad Corner: No. 126 ........................................ R.E. Woodrow 161

Book Review ................................................................. 170

Problems: 1651–1660 ....................................................... 171

Solutions: 93, 964, 1415, 1425, 1529, 1532–1541, 1543–1545 ................. 173
THE OLYMPIAD CORNER
No. 126
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This month we begin with the Canadian Mathematics Olympiad for 1991, which we reproduce with the permission of the Canadian Olympiad Committee of the Canadian Mathematical Society. My thanks to Ed Barbeau for sending me the contest along with the “official” solutions which will be given in the next issue.

1991 CANADIAN MATHEMATICS OLYMPIAD
April 1991
Time: 3 hours

1. Show that the equation \( x^2 + y^5 = z^3 \) has infinitely many solutions in integers \( x, y, z \) for which \( xyz \neq 0 \).

2. Let \( n \) be a fixed positive integer. Find the sum of all positive integers with the following property: in base 2, it has exactly \( 2n \) digits consisting of \( n \) 1’s and \( n \) 0’s. (The first digit cannot be 0.)

3. Let \( C \) be a circle and \( P \) a given point in the plane. Each line through \( P \) which intersects \( C \) determines a chord of \( C \). Show that the midpoints of these chords lie on a circle.

4. Ten distinct numbers from the set \( \{0, 1, 2, \ldots, 13, 14\} \) are to be chosen to fill in the ten circles in the diagram. The absolute values of the differences of the two numbers joined by each segment must be different from the values for all other segments. Is it possible to do this? Justify your answer.

5. In the figure, the side length of the large equilateral triangle is 3 and \( f(3) \), the number of parallelograms bounded by sides in the grid, is 15. For the general analogous situation, find a formula for \( f(n) \), the number of parallelograms, for a triangle of side length \( n \).

* * *
The next set of problems are from the twentieth annual United States of America Mathematical Olympiad, written in April. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588-0322. As always we welcome your original "nice" solutions and generalizations.

20th USA MATHEMATICAL OLYMPIAD
April 23, 1991
Time: 3.5 hours

1. In triangle $ABC$, angle $A$ is twice angle $B$, angle $C$ is obtuse, and the three side lengths $a$, $b$, $c$ are integers. Determine, with proof, the minimum possible perimeter.

2. For any nonempty set $S$ of numbers, let $\sigma(S)$ and $\pi(S)$ denote the sum and product, respectively, of the elements of $S$. Prove that

$$\sum \frac{\sigma(S)}{\pi(S)} = (n^2 + 2n) - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)(n + 1),$$

where the "$\sum$" denotes a sum involving all nonempty subsets $S$ of $\{1, 2, 3, \ldots, n\}$.

3. Show that, for any fixed integer $n \geq 1$, the sequence

$$2, 2^2, 2^3, 2^{2^2}, \ldots \mod n$$

is eventually constant. [The tower of exponents is defined by $a_1 = 2$, $a_{i+1} = 2^{a_i}$. Also, $a \mod n$ means the remainder which results from dividing $a$ by $n$.]

4. Let $a = \frac{m^{n+1} + n^{m+1}}{m^n + n^m}$, where $m$ and $n$ are positive integers. Prove that $a^m + a^n \geq m^m + n^n$. [You may wish to analyze the ratio $\frac{N^{-N}}{a^{-N}}$ for real $a \geq 0$ and integer $N \geq 1$.]

5. Let $D$ be an arbitrary point on side $AB$ of a given triangle $ABC$, and let $E$ be the interior point where $CD$ intersects the external, common tangent to the incircles of triangles $ACD$ and $BCD$. As $D$ assumes all positions between $A$ and $B$, prove that point $E$ traces the arc of a circle.

*   *   *
Next is a selection of problems from the 15th All Union Mathematical Olympiad—Tenth Grade. These appeared in Kvant and were translated by Hillel Gauchman and Duane Broline of Eastern Illinois University, Charleston. My thanks to them for sending these in.

15TH ALL UNION MATHEMATICAL OLYMPIAD—TENTH GRADE

First Day

1. Find natural numbers $a_1 < a_2 < \ldots < a_{2n+1}$ which form an arithmetic sequence such that the product of all terms is the square of a natural number. (5 points)

2. The numbers 1, 2, \ldots, $n$ are written in some order around the circumference of a circle. Adjacent numbers may be interchanged provided the absolute value of their difference is larger than one. Prove in a finite number of such interchanges it is possible to rearrange the numbers in their natural order. (5 points)

3. Let $ABC$ be a right triangle with right angle at $C$ and select points $D$ and $E$ on sides $AC$ and $BC$, respectively. Construct perpendiculars from $C$ to each of $DE$, $EA$, $AB$, and $BD$. Prove that the feet of these perpendiculars are on a single circle. (10 points)

4. Let $a \geq 0$, $b \geq 0$, $c \geq 0$, and $a + b + c \leq 3$. Prove

$$\frac{a}{1 + a^2} + \frac{b}{1 + b^2} + \frac{c}{1 + c^2} \leq \frac{3}{2} \leq \frac{1}{1 + a} + \frac{1}{1 + b} + \frac{1}{1 + c}.$$  
(first inequality 3 points, second inequality 7 points)

Second Day

5. Prove, for any real number $c$, that the equation

$$x(x^2 - 1)(x^2 - 10) = c$$

cannot have five integer solutions. (10 points)

6. A rook is placed on the lower left square of a chessboard. On each move, the rook is permitted to jump one square either vertically or horizontally. Prove the rook may be moved so that it is on one square once, one square twice, \ldots, one square 64 times, and

(a) so that the last move is to the lower left square,

(b) so that the last move is to a square adjacent (along an edge) to the lower left square.

(part (a) 4 points, part (b) 4 points)

7. Three chords, $AA_1$, $BB_1$, $CC_1$, to a circle meet at a point $K$ where the angles $B_1KA$ and $AKC_1$ are $60^\circ$, as shown. Prove that

$$KA + KB + KC = KA_1 + KB_1 + KC_1.$$  
(10 points)
8. Is it possible to put three regular tetrahedra, each having sides of length one, inside a unit cube so that the interiors of the tetrahedra do not intersect (the boundaries are allowed to touch)? (12 points)

*   *   *   *

We now turn to solutions of “archive” problems received from the readership. The problems are from the 1985 Spanish Olympiad that appeared in the May 1986 number of the Corner.


L and M are points on the sides AB and AC, respectively, of triangle ABC such that \( \frac{AL}{LB} = \frac{2AB}{5} \) and \( \frac{AM}{MC} = \frac{3AC}{4} \). If BM and CL intersect at P, and AP and BC intersect at N, determine \( \frac{BN}{NC} \).

Solution by Hans Engelhaupt, Gundelsheim, Germany.

From Ceva’s theorem one has

\[
\frac{AL}{LB} \cdot \frac{BN}{NC} \cdot \frac{CM}{MA} = 1
\]

This gives \( \frac{BN}{NC} = 9/2 \) and so \( \frac{BN}{BC} = 9/11 \).


Determine all the real roots of \( 4x^4 + 16x^3 - 6x^2 - 40x + 25 = 0 \).

Solution by Hans Engelhaupt, Gundelsheim, Germany.

Dividing by \( x^2 \) we get \( 4x^2 + 16x - 6 - 40/x + 25/x^2 = 0 \). The substitution \( z = 2x - 5/x \) yields \( z^2 + 8z + 14 = 0 \). This gives solutions \( z_1 = -4 + \sqrt{2} \) and \( z_2 = -4 - \sqrt{2} \).

Case 1. \( 2x^2 - (-4 + \sqrt{2})x - 5 = 0 \). Then

\[
x = \frac{-4 + \sqrt{2} \pm \sqrt{58 - 8\sqrt{2}}}{4}.
\]

Case 2. \( 2x^2 + (4 + \sqrt{2})x - 5 = 0 \). Then

\[
x = \frac{-4 - \sqrt{2} \pm \sqrt{58 + 8\sqrt{2}}}{4}.
\]

This gives the four real roots.


Let \( \mathbb{Z} \) be the set of integers and \( \mathbb{Z} \times \mathbb{Z} \) be the set of ordered pairs of integers. On \( \mathbb{Z} \times \mathbb{Z} \), define \( (a, b) + (a', b') = (a + a', b + b') \) and \( -(a, b) = (-a, -b) \). Determine if there exists a subset \( E \) of \( \mathbb{Z} \times \mathbb{Z} \) satisfying:

(i) Addition is closed in \( E \),

(ii) \( E \) contains \((0,0)\),

(iii) For every \( (a, b) \neq (0,0) \), \( E \) contains exactly one of \( (a, b) \) and \(- (a, b) \).
Yes. Let
\[ E = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : \text{ either } a \geq 0, b \geq 0 \text{ or } a < 0, b > 0\} \]
(i.e. \( E \) is the set of all lattice points in the first and second quadrants, together with those on the non-negative \( x \)-axis). It is easy to verify that \( E \) satisfies all three conditions.

Determine the value of \( p \) such that the equation \( x^5 - px - 1 = 0 \) has two roots \( r \) and \( s \) which are the roots of an equation \( x^2 - ax + b = 0 \) where \( a \) and \( b \) are integers.

Solution by Hans Engelhaupt, Gundelsheim, Germany, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.
If \( p \) satisfies the condition
\[ x^5 - px - 1 = (x^2 - ax + b)(x^3 + ux^2 + vx - 1/b), \]
comparing coefficients of \( x^4 \) we get \( u - a = 0 \), so \( u = a \). From coefficients of \( x^3 \) and \( x^2 \) we then get, respectively,
\[
\begin{align*}
&b - a^2 + v = 0 \quad \text{and} \quad -\frac{1}{b} - av + ab = 0, \\
&b - a^2 + v = 0 \quad \text{and} \quad -\frac{1}{b} - av + ab = 0,
\end{align*}
\]
and from coefficients of \( x \),
\[
\frac{a}{b} + bv = -p. 
\]
From (1),
\[
-\frac{1}{b} - a(a^2 - b) + ab = 0.
\]
Since \( a \) and \( b \) are integers it follows that \( b \) must be \( \pm 1 \).
Now, \( b = -1 \) gives \( a^3 + 2a - 1 = 0 \). Since \((-1)^3 + 2(-1) - 1 = -4 \) and \( 1^3 + 2 \cdot 1 - 1 = 2 \), this gives no rational and hence no integer solutions for \( a \). The case \( b = 1 \) gives \( a^3 - 2a + 1 = 0 \) which gives \( a = 1 \). From (1) \( v = 0 \) and so from (2) \( p = -1 \).

A square matrix is “sum-magic” if the sum of all the elements on each row, column, and major diagonal is constant. Similarly, a square matrix is “product-magic” if the product of all the elements in each row, column, and major diagonal is constant. Determine if there exist \( 3 \times 3 \) matrices of real numbers which are both “sum-magic” and “product-magic”.

Solution by Hans Engelhaupt, Gundelsheim, Germany.
If the square is “sum-magic” then it must have the form
\[
\begin{bmatrix}
m - x & m + x + y & m - y \\
m + x - y & m & m - x + y \\
m + y & m - x - y & m + x
\end{bmatrix}
\]
where $3m$ is the constant value of the sums. [Editor’s note. To see this, first observe that we can assume that the row sum is zero, hence $m = 0$, since subtracting the same amount from all entries gives another “sum-magic” matrix. We may then suppose that the matrix has the form

$$
\begin{bmatrix}
-x & x + y & -y \\
-a & a + b & -b \\
-x + a & -(x + y + a + b) & y + b
\end{bmatrix}.
$$

Now from the diagonal sums,

$$
a + 2b + y - x = 0 \quad \text{and} \quad 2a + b + x - y = 0.
$$

Adding these two equations gives $a = -b$, so $a = y - x$ and $b = x - y$, from which the claim is immediate.]

If the square is “product-magic” then from the diagonals,

$$(m - x)m(m + x) = (m - y)m(m + y). \tag{1}$$

**Case 1.** $m = 0$. From the row and column products,

$$xy(x + y) = 0 = xy(x - y).$$

Solving, we obtain $x = 0$ or $y = 0$ (or both). This gives the two forms

$$
\begin{bmatrix}
r & -r & 0 \\
-r & 0 & r \\
0 & r & -r
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
0 & s & -s \\
-s & 0 & s \\
s & -s & 0
\end{bmatrix}.
$$

**Case 2.** $m \neq 0$. Then from (1), $x^2 = y^2$. If $x = y$, then from the product for the second row and column,

$$m(m^2 - 4x^2) = m^3$$

so $m^2 - 4x^2 = m^2$ and $x = 0$, giving all entries equal. If $x = -y$ we similarly obtain $m^3 = m(m^2 - 4x^2)$ and $x = 0$ giving all entries equal.

In summary, a “sum-magic” square that is also “product-magic” has all entries equal unless one of the diagonals contains only zeroes, and then the matrix is uniquely determined by an off-diagonal entry.

$$*
\quad *
\quad *
$$

We now turn to solutions to the problems from the October 1989 number of the Corner. These were problems proposed to the jury, but not used, at the 30th IMO at Braunschweig, (then) West Germany [1989: 225–226]. We give the solutions we’ve received which differ from the “official” solutions published in the booklet *30th International Mathematical Olympiad* (see the review on [1991: 42]).
6. Proposed by Greece.
Let \( g : \mathbb{C} \to \mathbb{C}, \omega \in \mathbb{C}, a \in \mathbb{C}, \) with \( \omega^3 = 1 \) and \( \omega \neq 1. \) Show that there is one and only one function \( f : \mathbb{C} \to \mathbb{C} \) such that
\[
f(z) + f(\omega z + a) = g(z), \quad z \in \mathbb{C}.
\]
Find the function \( f.\)

Solutions by George Evagelopoulos, Athens, Greece, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.
If \( f : \mathbb{C} \to \mathbb{C} \) satisfies
\[
g(z) = f(z) + f(\omega z + a) \quad \text{for all} \quad z \in \mathbb{C}, \tag{1}
\]
then
\[
g(\omega z + a) = f(\omega z + a) + f(\omega^2 z + \omega a + a) \quad \text{for all} \quad z \in \mathbb{C}. \tag{2}
\]
Since \( \omega^3 = 1 \) and \( 1 + \omega + \omega^2 = 0, \) we have
\[
\omega^2 z + \omega a + a = \omega^2 z + a(1 + \omega) = \omega^2 (z - a) = \frac{z - a}{\omega},
\]
and so
\[
g\left(\frac{z - a}{\omega}\right) = f\left(\frac{z - a}{\omega}\right) + f\left(\omega \cdot \frac{z - a}{\omega} + a\right) = f(\omega^2 z + \omega a + a) + f(z). \tag{3}
\]
Therefore (1) minus (2) plus (3) yields
\[
g(z) - g(\omega z + a) + g\left(\frac{z - a}{\omega}\right) = 2f(z),
\]
and so
\[
f(z) = \frac{1}{2} \left\{ g(z) - g(\omega z + a) + g\left(\frac{z - a}{\omega}\right) \right\}.
\]
On the other hand, it is easily checked that this choice of \( f \) satisfies (1) and hence it is the unique function with this property.

7. Proposed by Hungary.
Define the sequence \( \{a_n\}_{n=1}^{\infty} \) of integers by
\[
\sum_{d|n} a_d = 2^n.
\]
Show that \( n|a_n. \) [Editor’s note. Of course \( x|y \) means that \( x \) divides \( y. \)]
Solution by Graham Denham, student, University of Alberta, and by Curtis Cooper, Central Missouri State University.

The proof is by induction on \( n \). For \( a_1 \), the result is obvious. Now assume that \( i | a_i \) for all \( i < n \). In order to show that \( n | a_n \), it will suffice to show that \( p^e | a_n \), where \( p^e \) is any prime-power divisor of \( n \). Say \( n = p^e k \), where \( k \) is not divisible by \( p \). Then

\[
2^n = \sum_{d \mid p^ek} a_d = \sum_{d \mid p^{e-1}k} a_d + \sum_{d \mid k} a_{p^e d} = 2^{p^{e-1}k} + \sum_{d \mid k, d < k} a_{p^e d}.
\]

So,

\[
a_{p^e k} = 2^{p^ek} - 2^{p^{e-1}k} - \sum_{d \mid k, d < k} a_{p^e d}.
\]

Now, by hypothesis, each term \( a_{p^e d} \) is divisible by \( p^e \), since \( p^e d < n \). Therefore,

\[
a_n = a_{p^e k} \equiv \left(2^{p^ek} - 2^{p^{e-1}k}\right) \mod p^e
\]

\[
\equiv 2^{p^{e-1}k} \left(2^{p^{e-1}k(p-1)} - 1\right) \mod p^e.
\]

If \( p = 2 \), then \( 2^{p^{e-1}k} \equiv 0 \mod p^e \), since \( p^e k \leq n \). For \( p > 2 \), we may use Euler’s Theorem, giving

\[
2^{\phi(p^e)} \equiv 1 \mod p^e,
\]

that is,

\[
2^{p^{e-1}(p-1)} \equiv 1 \mod p^e.
\]

So \( 2^{p^{e-1}(p-1)k} \equiv 1 \mod p^e \), again giving \( a_n \equiv 0 \mod p^e \). Hence \( p^e | a_n \), and as \( p \) was arbitrary, \( n | a_n \), and the result follows by induction.

[Editor’s note. This is the solution sent in by Graham Denham. The solution of Curtis Cooper differed by an initial application of Möbius inversion giving \( a_n = \sum_{d \mid n} \mu(d) 2^n/d \), where \( \mu \) denotes the Möbius function. For an elegant combinatorial solution see the official solution, in which \( a_n \) is viewed as the number of nonrepeating 0–1 sequences.]


Let \( a, b, c, d, m, n \) be positive integers such that

\[
a^2 + b^2 + c^2 + d^2 = 1989,
\]

\[
a + b + c + d = m^2,
\]

and the largest of \( a, b, c, d \) is \( n^2 \). Determine, with proof, the values of \( m \) and \( n \).

Solution by Hans Engelhaupt, Gundelsheim, Germany, and also by George Evagelopoulos, Athens, Greece.

Without loss of generality we may suppose \( 0 < a \leq b \leq c \leq d \). Let \( S = a^2 + b^2 + c^2 + d^2 \). Since \( x^2 + y^2 \geq 2xy \) it follows that

\[
3S \geq 2(ab + ac + ad + bc + bd + cd)
\]
whence \[ 4S \geq (a + b + c + d)^2. \]

Thus \(4 \cdot 1989 \geq m^4\) and \(m \leq 9\).

But \((a+b+c+d)^2 > a^2 + b^2 + c^2 + d^2\) so \(m^4 > 1989\) and \(m > 6\). Since \(a^2 + b^2 + c^2 + d^2\) is odd, so is \(m^2 = a + b + c + d\), thus \(m = 7\) or \(m = 9\). Suppose for a contradiction that \(m = 7\). Now \((49 - d)^2 = (a + b + c)^2 > a^2 + b^2 + c^2 = 1989 - d^2\). Thus \(d^2 - 49d + 206 > 0\). It follows that \(d > 44\) or \(d \leq 4\). However, \(d < 45\), since \(45^2 = 2025\). Also, \(d \leq 4\) implies \(a^2 + b^2 + c^2 + d^2 \leq 64 < 1989\).

It follows that \(m = 9\). Now \(n^2 = d > 16\), since \(d \leq 16\) implies \(a + b + c + d \leq 64 < 81\). As before \(d \leq 44\) so \(n^2 = 25\) or \(36\). If \(d = n^2 = 25\) then let \(a = 25 - p\), \(b = 25 - q\), \(c = 25 - r\), with \(p, q, r \geq 0\). Furthermore \(a + b + c = 56\) implies \(p + q + r = 19\), \((25 - p)^2 + (25 - q)^2 + (25 - r)^2 = 1364\) implies \(p^2 + q^2 + r^2 = 439\). Now \((p + q + r)^2 > p^2 + q^2 + r^2\) gives a contradiction.

Thus the only possibility is that \(n = 6\), and there is a solution with \(a = 12\), \(b = 15\), \(c = 18\), \(d = 36\). Thus \(m = 9\) and \(n = 6\).

11. Proposed by Mongolia.

Seven points are given in the plane. They are to be joined by a minimal number of segments such that at least two of any three points are joined. How many segments has such a figure? Give an example.

Generalization and solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We use the terminology of graph theory. For each positive integer \(n\), let \(\ell_n\) denote the minimum number of edges that a graph on \(n\) vertices must have so that among any three vertices at least two are joined by an edge. Then we claim

\[
\ell_n = \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor.
\]

To see this it is useful to consider \(m_n\), the maximum number of edges that a graph on \(n\) vertices can have, but contain no triangle. We first show that \(\ell_n + m_n = \binom{n}{2}\). Let \(G\) be a graph on \(n\) vertices with \(\ell_n\) edges so that there are no "empty triangles". Then the complementary graph \(\overline{G}\), obtained by interchanging edges and non-edges, is a graph without triangles, so \(\binom{n}{2} - \ell_n \leq m_n\). Similarly one argues that \(\ell_n \leq \binom{n}{2} - m_n\).

Now by Turán’s theorem \(m_n = \lfloor n^2/4 \rfloor\) (c.f. Ex. 30, p. 68 of Combinatorial Problems and Exercises by L. Lovász). Thus \(\ell_n = \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor\). By checking separately \(n\) even and \(n\) odd, one obtains \(\ell_n = \lfloor (n - 1)^2/4 \rfloor\) in all cases. This situation is realized by the disjoint union of two complete graphs on \((n - 1)/2\) and \((n + 1)/2\) points respectively if \(n\) is odd, and by two copies of the complete graph \(K_{n/2}\) if \(n\) is even.

* * *

This completes the correct solutions received for problems from the October 1989 number and the space we have this month. Send me your problems and nice solutions!
BOOK REVIEW


This book was put together by the Chinese Mathematical Olympiad Committee as a gift to the Leaders and Deputy Leaders of the teams participating in the 31st International Mathematical Olympiad in Beijing. It consists of four parts.

The first part consists of an essay sketching the history and development of the Olympiad movement in China, climaxing in the 1990 IMO. The second part consists of eight articles on mathematics that arose from the Olympiads. Some are translations of articles of exceptional merit that were published previously in Chinese, while others are contributed especially for this volume.

Here is a problem considered in the article "A conjecture concerning six points in a square" by L. Yang and J. Zhang, translated from the Chinese version published in 1980. Everyone knows that 9 points in a square of area 1 are sufficient to guarantee that 3 of them will determine a triangle of area at most 1/4. Actually, 9 points are not necessary. What is the minimum?

The answer is 6. A key lemma states that among 4 points determining a convex quadrilateral in a triangle of area 1, 3 will determine a triangle of area at most 1/4. The simple proof is an elegant combination of Euclidean geometry and the Arithmetic-Mean Geometric-Mean Inequality.

Another easy corollary of this lemma is that 5 points in a triangle of area 1 are sufficient to guarantee that 3 of them will determine a triangle of area at most 1/4. Here, it is easy to see that 5 points are also necessary.

The third part of the book consists of three articles on the art and science of problem-proposing and problem-solving. The fourth part consists of the problems and solutions of the first five annual Mathematical Winter Camps in China.

A very limited number of extra copies of this book have been printed. The price at US $5.80 (plus 25% overseas postage) is a true bargain. It may be ordered with prepayment from:

Hunan Education Publishing House,
1 Dongfen Road, Changsha, Hunan,
410005, People’s Republic of China.
PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1992, although solutions received after that date will also be considered until the time when a solution is published.

1651. Proposed by George Tsintsifas, Thessaloniki, Greece.
Let $ABC$ be a triangle and $A_1, B_1, C_1$ the common points of the inscribed circle with the sides $BC, CA, AB$, respectively. We denote the length of the arc $B_1C_1$ (not containing $A_1$) of the incircle by $S_a$, and similarly define $S_b$ and $S_c$. Prove that

$$\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c} \geq \frac{9\sqrt{3}}{\pi}.$$

1652. Proposed by Murray S. Klamkin, University of Alberta.
Given fixed constants $a, b, c > 0$ and $m > 1$, find all positive values of $x, y, z$ which minimize

$$\frac{x^m + y^m + z^m + a^m + b^m + c^m}{6} - \left(\frac{x + y + z + a + b + c}{6}\right)^m.$$

1653. Proposed by Toshio Seimiya, Kawasaki, Japan.
Let $P$ be the intersection of the diagonals $AC, BD$ of a quadrangle $ABCD$, and let $M, N$ be the midpoints of $AB, CD$, respectively. Let $l, m, n$ be the lines through $P, M, N$ perpendicular to $AD, BD, AC$, respectively. Prove that if $l, m, n$ are concurrent, then $A, B, C, D$ are concyclic.

1654*. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.
Let $x, y, z$ be positive real numbers. Show that

$$\sum \frac{x}{x + \sqrt{(x + y)(x + z)}} \leq 1,$$

where the sum is cyclic over $x, y, z$, and determine when equality holds.
1655. Proposed by Jordi Dou, Barcelona, Spain.
Let $ABCD$ be a trapezoid with $AD \parallel BC$. $M, N, P, Q, O$ are the midpoints of $AB, CD, AC, BD, MN$, respectively. Circles $m, n, p, q$ all pass through $O$, and are tangent to $AB$ at $M$, to $CD$ at $N$, to $AC$ at $P$, and to $BD$ at $Q$, respectively. Prove that the centres of $m, n, p, q$ are collinear.

1656. Proposed by Hidetosi Fukagawa, Aichi, Japan.
Given a triangle $ABC$, we take variable points $P$ on segment $AB$ and $Q$ on segment $AC$. $CP$ meets $BQ$ in $T$. Where should $P$ and $Q$ be located so that the area of $\Delta PQT$ is maximized?

Pythagoras the eternal traveller reached Brahmapura on one of his travels. “Dear Pythagoras”, told the townspeople of Brahmapura, “there on the top of that tall vertical mountain $99$ Brahmis away resides Brahmagupta, who exhibits unmatched skills in both travel through the air and mathematics”. “Aha”, exclaimed Pythagoras, “I must have a chat with him before leaving.” Pythagoras made his way directly to the foot of the mountain. No sooner had he reached it than he found himself in a comforting magic spell that flew him effortlessly to the summit where Brahmagupta received him. The two exchanged greetings and ideas. “You will have noticed that the foot and summit of this mountain forms with the town a triangle that is Pythagorean [an integer-sided right triangle—Ed.]”, Brahmagupta remarked. “I can rise high vertically from the summit and then proceed diagonally to reach the town thus making yet another Pythagorean triangle and in the process equalling the distance covered by you from the town to the summit.” Tell me, dear Crux problem solver, how high above the mountain did Brahmagupta rise?

1658. Proposed by Avinoam Freedman, Teaneck, New Jersey.
Let $P$ be a point inside circle $O$ and let three rays from $P$ making angles of $120^\circ$ at $P$ meet $O$ at $A, B, C$. Show that the power of $P$ with respect to $O$ is the product of the arithmetic and harmonic means of $PA, PB$ and $PC$.

1659°. Proposed by Stanley Rabinowitz, Westford, Massachusetts.
For any integer $n > 1$, prove or disprove that the largest coefficient in the expansion of

$$(1 + 2x + 3x^2 + 4x^3)^n$$

is the coefficient of $x^{2n}$.

1660. Proposed by Isao Ashiba, Tokyo, Japan.
Construct equilateral triangles $A'B'C, B'C'A, C'A'B$ exterior to triangle $ABC$, and take points $P, Q, R$ on $AA', BB', CC'$, respectively, such that

$$\frac{AP}{AA'} + \frac{BQ}{BB'} + \frac{CR}{CC'} = 1.$$ 

Prove that $\Delta PQR$ is equilateral.

* * * * *
SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


Is there a convex polyhedron having exactly seven edges?

V. Solution by William Pippin and James Underwood, students, The Ohio State University.

Since the vertices and edges of a polyhedron form the vertices and edges of a simple graph in which each vertex has degree at least 3, the following result provides an immediate negative answer.

A simple graph in which each vertex has degree at least 3 cannot have exactly 7 edges.

Proof. Suppose that the graph has V vertices. Then the number of edges must be at least $3V/2$, since each vertex is incident to at least 3 edges and each edge has 2 ends. But if $V \geq 5$, then $3V/2 \geq 15/2 > 7$. On the other hand, if $V \leq 4$ then the graph cannot have more than $\binom{4}{2} = 6$ edges.


Let $T$ be the image of the Euler $\phi$-function, that is,

$$T = \{\phi(n) : n = 1, 2, 3, \ldots\}.$$

Prove or disprove that $T$ is a Dirichlet set, as defined in the proposer’s article “Elementary Dirichlet sets” [1984: 206–209, esp. p. 206 and last paragraph p. 209].

Solution by Sam Maltby, student, University of Calgary.

We need to show that if $S$ is an arithmetic sequence,

$$S = \{a, a + d, a + 2d, a + 3d, \ldots\}$$

with $a$ and $d$ positive integers, then $|S \cap T| \in \{0, 1, \infty\}$. We in fact prove: if $S \cap T$ contains an element greater than or equal to $d$, then it contains infinitely many elements.

Assume $t \in S \cap T$ with $t \geq d$. Since $t \in T$,

$$t = \phi(m) = p_1^{\alpha_1}(p_1 - 1)p_2^{\alpha_2}(p_2 - 1)\ldots p_r^{\alpha_r}(p_r - 1)$$

(1)

where $m = p_1^{\alpha_1+1}p_2^{\alpha_2+1}\ldots p_r^{\alpha_r+1}$, the $p_i$’s are distinct primes and each $\alpha_i \geq 0$.

Now if we can find infinitely many elements of $T$ congruent to $a$ modulo $d$, we are done. Since $t \equiv a \pmod{d}$, it is sufficient to find these elements congruent to $t$ mod $d$.

Let

$$d = kp_1^\beta_1 p_2^\beta_2 \ldots p_r^\beta_r, \quad \beta_i \geq 0 \text{ for all } i, \quad (k, p_i) = 1 \text{ for all } i.$$
We cannot have \( \beta_i > \alpha_i \) for all \( i \), for then we would have
\[
d \geq p_1^\beta_1 \ldots p_r^\beta_r \geq p_1^{\alpha_1+1} \ldots p_r^{\alpha_r+1} > t .
\]
So \( \beta_i \leq \alpha_i \) for some \( i \), say \( \beta_r \leq \alpha_r \).

Put \( x = tp_r^{n_0(d)} \), where \( n \) is any positive integer; then I claim that
\[
x \equiv t \mod d
\]
for all \( n = 1, 2, \ldots \). Putting
\[
d = k'p_r^{\beta_r} \quad (\text{i.e., } k' = k_1^{\beta_1} \ldots p_{r-1}^{\beta_{r-1}}) ,
\]
we get from (1) that \( t \equiv 0 \mod p_r^{\beta_r} \), so
\[
x \equiv t \equiv 0 \mod p_r^{\beta_r} .
\]
Also by (2) and Euler’s theorem,
\[
x = t \cdot p_r^{\omega(k')n_0(p_r^{\beta_r})} \equiv t \cdot 1^{n_0(p_r^{\beta_r})} \mod k' \equiv t \mod k',
\]
so since \((k', p_r^{\beta_r}) = 1\) we have \( x \equiv t \mod d \), as claimed. Thus \( x \in S \) for all \( n \).

Also, \( x \) is of the form (1), so \( x \in T \). Thus \( T \) is an elementary Dirichlet set.

Note incidentally that the sequence \( S = \{1, 3, 5, 7, \ldots \} \), with and without its first element, shows that \( |S \cap T| \) may equal 0 or 1.

* * * * *


Given the system of differential equations
\[
\begin{align*}
\dot{x}_1 &= -(c_{12} + c_{13})x_1 + c_{12}x_2 + c_{13}x_3 \\
\dot{x}_2 &= c_{21}x_1 - (c_{21} + c_{23})x_2 + c_{23}x_3 \\
\dot{x}_3 &= c_{31}x_1 + c_{32}x_2 - (c_{31} + c_{32})x_3 ,
\end{align*}
\]
where the \( c \)'s are positive constants, show that \( \lim_{t \to \infty} x_i(t) \) is a weighted average, independent of \( i \), of the initial values \( x_1(0) \), \( x_2(0) \), \( x_3(0) \).

II. Comment by the editor.

It has been pointed out by John Lindsey, Northern Illinois University, Dekalb, that an “explanation” offered by the editor for one point in the published proof was incorrect! On [1990: 115], in an “editor’s note” appears the equation
\[
x_i(t) = x_i(0) + f_i(t)e^{mt} .
\]
This equation, and the rest of the editor’s note, should be ignored! Instead one can substitute the following argument, obtained by consultation with the solution’s author, Kee-Wai Lau.
From equation (2) on [1990: 115],
\[ x_i(t) = K + e_i e^{mt} + f_i t e^{mt} \]
so that
\[ \dot{x}_i(t) = e_i m_1 e^{mt} + f_i e^{mt} + f_i m_1 t e^{mt} \, . \]
Thus from \( \dot{x}_i(0) = 0 \) we get \( e_i m_1 + f_i = 0 \), and since \( e_1 = e_2 = e_3 \) it follows that \( f_1 = f_2 = f_3 \). Hence \( x_1(t) = x_2(t) = x_3(t) \), and we deduce from the original system that \( \dot{x}_1(t) = \dot{x}_2(t) = \dot{x}_3(t) = 0 \). Thus \( x_1(t) = x_2(t) = x_3(t) = \) constant, as claimed. Apologies to Professor Lau.

Lindsey also contributed a generalization of this problem to \( n \) variables.

\[
\begin{array}{cccccc}
\ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]


Let \( D \) be the midpoint of side \( BC \) of the equilateral triangle \( ABC \) and \( \omega \) a circle through \( D \) tangent to \( AB \), cutting \( AC \) in points \( B_1 \) and \( B_2 \). Prove that the two circles, distinct from \( \omega \), which pass through \( D \) and are tangent to \( AB \), and which respectively pass through \( B_1 \) and \( B_2 \), have a point in common on \( AC \).

III. Comment by Chris Fisher, University of Regina.

On [1990: 148] the editor asked for another solution to this problem. Ironically \( Crux \) has already published several as solutions to \( Crux \) 975 [1985: 328–331]. Note that the figure in the comment by Michal Szurek and me [1985: 330] is the same as the one Jordi Dou used on [1990: 148]. If you invert that figure you get (according to Dou) \( Crux \) 1425; if you project it (as explained in our comment) you get \( Crux \) 975. In other words, both problems are specializations of the classical theorem known as Poncelet’s Porism.

A further comment on this problem was received from P. PENNING, Delft, The Netherlands.

\[
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\end{array}
\]


Given points \( A, B, C \) on line \( l \) and \( A', B', C' \) on line \( l' \), construct the points \( P \neq l \cap l' \) such that the three angles \( APA', B PB', CPC' \) have the same pair of bisectors.

Solution by the proposer.

Let \( \pi \) be the projectivity of \( l \) onto \( l' \) defined by \( \pi(A) = A', \pi(B) = B', \pi(C) = C' \).

Let \( \omega \) be the conic tangent to \( l, l', AA', BB' \) and \( CC' \). We put \( l \cap l' = S = T' \), \( \pi(S) = S' \), and \( \pi^{-1}(T') = T \); note that \( S' \) and \( T \) are the points where \( l' \) and \( l \) meet \( \omega \).

For a point \( X \), let \( \pi_X \) denote the projectivity of the pencil of lines through \( X \) that takes \( XA, XB, XC \) to \( XA', XB', XC' \). We seek the position of \( X = P \) for which \( \pi_P \) is induced by a reflection of the plane; in other words, \( \pi_P \) is an involution whose fixed lines are orthogonal. For \( \pi_X \) to be an involution it is necessary and sufficient that \( X \) be on the line \( S'T \) (because \( \pi_X(XS) = XS' \) must coincide with \( \pi_X^{-1}(XT') = XT \)).
The fixed lines of \( \pi \) are the tangents from \( X \) to \( \omega \). In order that these be perpendicular it is necessary and sufficient that \( X \) be on the circle \( \mu \) (of Monge) that is the locus of points from which the tangents to \( \omega \) form a right angle.

Therefore the required points \( P \) are the points of intersection of \( ST' \) with \( \mu \).

Construction of \( P \). Since \( ST' \) is the axis of \( \pi \) it must contain \( D = AC' \cap A'C' \) and \( E = AB' \cap A'B' \) (as well as \( BC' \cap B'C' \)). But \( ST' \) is also the polar of \( S \) with respect to \( \omega \); thus if \( M \) is the midpoint of \( ST' \), then \( SM \) passes through the centre \( O \) of \( \omega \) (which is also the centre of \( \mu \)). Analogously, letting \( F = AA' \cap BB' \), \( G = A'B' \cap FS \) and \( R = GT \cap AA' \), then \( RT \) is the polar of \( A \). Letting \( N \) be the midpoint of \( RT \), \( AN \) passes through \( O \). Thus \( O = SM \cap AN \). It remains to find a point on the circle \( \mu \). Let \( A'', B'', T'' \) be the orthogonal projections of \( A', B', T' \) on \( l \) (note \( T'' = T' \)). Let \( U \) and \( U' \) be the fixed points of the projectivity \( A, B, T \to A'', B'', T'' \). Then the perpendiculars to \( l \) through \( U, U' \) are tangent to \( \omega \) and therefore \( U \) and \( U' \) lie on \( \mu \). (The fixed points of a projectivity are easily found; for example, project \( A, B, T, A'', B'', T'' \) onto any convenient circle \( \alpha \) from a point \( V \) of \( \alpha \). Let \( a, b, t, a'', b'', t'' \) be these projections onto \( \alpha \). The line through points \( at'' \cap a''t \) and \( bt'' \cap b''t \) cuts \( \alpha \) in \( u \) and \( u' \), which project back to give us \( U \) and \( U' \).)

\[ \boxed{\begin{array}{c}
\end{array}} \]


Determine all (possibly degenerate) triangles \( ABC \) such that

\[
(1 + \cos B)(1 + \cos C)(1 - \cos A) = 2 \cos A \cos B \cos C.
\]

Solution by Kee-Wai Lau, Hong Kong.

We show that the equality of the problem holds if and only if \( A = 0 \) and \( B = C = \pi/2 \).

Since

\[
2 \cos A \cos B \cos C = (1 + \cos B)(1 + \cos C)(1 - \cos A) \geq 0,
\]

so \( 0 \leq A, B, C \leq \pi/2 \). Clearly \( A \neq \pi/2 \), and if \( A = 0 \) then \( B = C = \pi/2 \). In what follows we assume that \( 0 < A < \pi/2 \) and \( 0 \leq B, C < \pi/2 \). For \( 0 \leq x < \pi/2 \), let \( f(x) = \ln(\sec x + 1) \). We have

\[
f''(x) = \frac{1 + \cos x - \cos^2 x}{(1 + \cos x) \cos^2 x} > 0
\]
and so \( f(x) \) is convex. Hence

\[
(\sec B + 1)(\sec C + 1)(\sec A - 1) \geq \left( \sec \frac{B + C}{2} + 1 \right)^2 (\sec A - 1)
\]

\[
= \left( \csc \frac{A}{2} + 1 \right)^2 (\sec A - 1)
\]

\[
> \csc^2 \frac{A}{2} (\sec A - 1)
\]

\[
= 2 \sec A \geq 2.
\]

Therefore

\[
(1 + \cos B)(1 + \cos C)(1 - \cos A) > 2 \cos A \cos B \cos C.
\]

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; and the proposer.

Festraets-Hamoir and the proposer solved the problem by showing that the given equality is equivalent to the orthocentre of the triangle lying on the excircle to side \( a \), which can be seen geometrically to happen only for the degenerate \((0^\circ, 90^\circ, 90^\circ)\) triangle.

* * * *


For any integers \( n \geq k \geq 0, n \geq 1 \), denote by \( p(n,k) \) the probability that a randomly chosen permutation of \( \{1, 2, \ldots, n\} \) has exactly \( k \) fixed points, and let

\[
P(n) = p(n, 0)p(n, 1)\ldots p(n, n).
\]

Prove that

\[
P(n) \leq \exp(-2^n n!).
\]

Solution.

Since \( p(n, n-1) = 0 \), \( P(n) = 0 \).

Solved by JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer, who claims he didn’t invent the problem, but that it was “circulating”.

This problem was intended as an “April Fool’s” joke by the editor, and, who knows, might have been better appreciated by the readers if the April 1990 Crux had been available anywhere near the beginning of April! Some future spring, when Crux is on schedule, the editor may be moved to try again.

* * * *
Triangle $H_1 H_2 H_3$ is formed by joining the feet of the altitudes of an acute triangle $A_1 A_2 A_3$. Prove that

$$\frac{s}{r} \leq \frac{s'}{r'},$$

where $s, s'$ and $r, r'$ are the semiperimeters and inradii of $A_1 A_2 A_3$ and $H_1 H_2 H_3$ respectively.

**Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.**

This will be a solution “from the books”. First we observe that if $S$ is the area of the triangle $ABC$, then $S = sr$, $s' = S/R$ and $r' = 2R \cos A_1 \cos A_2 \cos A_3$ (see p. 191 of Johnson, Advanced Euclidean Geometry). Then the proposed inequality can be written

$$\frac{s}{r} \leq \frac{sr}{2R^2 \cos A_1 \cos A_2 \cos A_3},$$

or equivalently

$$\cos A_1 \cos A_2 \cos A_3 \leq \frac{r^2}{2R^2},$$

and this last is an inequality of W.J. Blundon (see problem E1925 of the American Math. Monthly, solution on pp. 196-197 of the February 1968 issue).

**Also solved by** NIELS BEJLEGAARD, Stavanger, Norway; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; L.J. HUT, Groningen, The Netherlands; WALThER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; MURRAY S. KLAMKin, University of Alberta; MARCiN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; BOB PRIELiPP, University of Wisconsin–Oshkosh; and the proposer.

Several solvers reduced the problem of one of the equivalent inequalities

$$(1 - \cos A_1)(1 - \cos A_2)(1 - \cos A_3) \geq \cos A_1 \cos A_2 \cos A_3,$$

which is contained in Crux 836 [1984: 228], or

$$4R^2 + 4Rr + 3r^2 \geq s^2,$$

which is item 5.8 of Bottema et al, Geometric Inequalities. See the proof of Crux 564 [1981: 150–153] for yet more equivalent forms. Also see the proof of Crux 1539, this issue!

Janous used item 35, p. 246 of Mitrović et al, Recent Advances in Geometric Inequalities, to refine the given inequality:

$$\frac{s'}{r'} = \frac{F}{2R^2 \cos A_1 \cos A_2 \cos A_3} = \tan A_1 \tan A_2 \tan A_3 \geq \frac{2R^2}{4F} = \frac{2R^2}{4s^2} \cdot \frac{s}{r},$$

this being a refinement because $3\sqrt{3}R \geq 2s$ (item 5.3 of Bottema et al).

Klamkin considers replacing $H_1 H_2 H_3$ by the pedal triangle of a point $P$, and asks for which $P$ the inequality still holds. He notes that it holds if $P$ lies on the circumcircle of $A_1 A_2 A_3$.

* * * * * *
Let \( P \) be a variable point inside an ellipse with equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]

Through \( P \) draw two chords with slopes \( b/a \) and \(-b/a\) respectively. The point \( P \) divides these two chords into four pieces of lengths \( d_1, d_2, d_3, d_4 \). Prove that \( d_1^2 + d_2^2 + d_3^2 + d_4^2 \) is independent of the location of \( P \) and in fact has the value \( 2(a^2 + b^2) \).

I. Solution by Richard I. Hess, Rancho Palos Verdes, California.

Let \( x' = (b/a)x \). Then the ellipse becomes the circle \( x'^2 + y^2 = \bar{b}^2 \) when plotted in \( x', y \) axes. Moreover the chords become at angles of 45° to the axes.

Rotate the circle so that the given chords are parallel to the axes. Denote the resulting coordinates of \( P \) by \((r, s)\) and the lengths of the pieces of chords through \( P \) by \( d_1', d_2', d_3', d_4' \). Then it’s clear that

\[
d_1' = \sqrt{\bar{b}^2 - s^2} - r \quad \text{and} \quad d_3' = \sqrt{\bar{b}^2 - s^2} + r,
\]

so

\[
d_1'^2 + d_3'^2 = 2\bar{b}^2 - 2s^2 + 2r^2.
\]

Similarly

\[
d_2' = \sqrt{\bar{b}^2 - r^2} - s \quad \text{and} \quad d_4' = \sqrt{\bar{b}^2 - r^2} + s,
\]

so

\[
d_2'^2 + d_4'^2 = 2\bar{b}^2 - 2r^2 + 2s^2.
\]

Thus

\[
d_1'^2 + d_2'^2 + d_3'^2 + d_4'^2 = 4\bar{b}^2,
\]

independent of the placement of \( P \).

In going from \((x', y)\)-coordinates to \((x, y)\)-coordinates as shown on the next page, we have

\[
d_i^2 = \left(1 + \frac{a^2}{\bar{b}^2}\right)x_i^2 = \left(1 + \frac{a^2}{\bar{b}^2}\right) \cdot \frac{1}{2}d_i'^2,
\]

and so by \((1)\),

\[
d_1^2 + d_2^2 + d_3^2 + d_4^2 = \frac{1}{2} \left(1 + \frac{a^2}{\bar{b}^2}\right)4\bar{b}^2 = 2(a^2 + b^2).
\]
II. Solution par Jacques Choné, Lycée Blaise Pascal, Clermont-Ferrand, France.

Soit \( (\alpha, \beta) \) les coordonnées de \( P \), avec \( \alpha^2/a^2 + \beta^2/b^2 - 1 \leq 0 \). Un vecteur unitaire des cordes considérées a pour coordonnées:

\[
\left( \frac{a}{\sqrt{a^2 + b^2}}, \frac{\varepsilon b}{\sqrt{a^2 + b^2}} \right), \quad \varepsilon \in \{-1, 1\}.
\]

Les points des cordes considérées ont pour coordonnées:

\[
\left( \alpha + \frac{\lambda a}{\sqrt{a^2 + b^2}}, \beta + \frac{\lambda \varepsilon b}{\sqrt{a^2 + b^2}} \right), \quad \lambda \text{ réel}.
\]

Les réels \( d_i \ (i = 1, \ldots, 4) \) sont les valeurs absolues des solutions de l’équation

\[
\frac{1}{a^2} \left( \alpha + \frac{\lambda a}{\sqrt{a^2 + b^2}} \right)^2 + \frac{1}{b^2} \left( \beta + \frac{\lambda \varepsilon b}{\sqrt{a^2 + b^2}} \right)^2 = 1,
\]

c’est à dire de

\[
\frac{2}{a^2 + b^2} \lambda^2 + \frac{2\lambda}{\sqrt{a^2 + b^2}} \left( \frac{\alpha}{a} + \frac{\varepsilon \beta}{b} \right) + \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 = 0.
\]

La somme des carrés des solutions de l’équation \( Ax^2 + Bx + C = 0 \) étant \((-B/A)^2 - 2C/A\), on obtient

\[
d_1^2 + d_2^2 + d_3^2 + d_4^2 = \left[ \left( \frac{\alpha}{a} + \frac{\beta}{b} \right)^2 + \left( \frac{\alpha}{a} - \frac{\beta}{b} \right)^2 \right] (a^2 + b^2) - 2 \left( \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 \right) (a^2 + b^2) = 2(a^2 + b^2),
\]

ce qui est le résultat désiré.

*Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona,*
Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALther JANOUS, Ursulinegymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIn E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Klamkin and Kuczma note that equation (1), the special case for circles, has already appeared in Crux on [1989: 293].

\[ \begin{align*}
\text{1536}^* & \text{. [1990: 109] Proposed by Walther Janous, Ursulinegymnasium, Innsbruck, Austria.}
\end{align*} \]

This being problem 1536 in volume 16 of \textit{Crux}, note that the numbers 16 and 1536 have the following nice property:

\[
\begin{array}{r}
16 \\
\end{array}
\begin{array}{r}
1536 \\
144 \\
96 \\
96 \\
0
\end{array}
\]

\[
\left(\begin{array}{c}
\text{in Europe: } 1536 : 16 = 96 \\
96 \\
0
\end{array}\right).
\]

That is, 1536 is exactly divisible by 16, and upon dividing 16 into 1536 via “long division” the last nonzero remainder in the display is equal to the quotient 1536/16 = 96. Assuming \textit{Crux} continues to publish 100 problems each year, what will be the next \textit{Crux} volume and problem numbers to have the same property?

\textit{Solution by Kenneth M. Wilke, Topeka, Kansas.}

Since there are exactly 100 problems published annually we have

\[ P = 100(v - 1) + p \]  \hspace{1cm} (1)

where \( p \) denotes the last two digits of the problem number \( P \) and \( v \) is the volume number. Both \( v \) and \( p \) are integers with \( v > 0 \) and \( 1 \leq p \leq 99 \). [This omits the last problem in each volume, whose number ends in 00, but it is easy to see such problems can never yield a solution.] Letting \( q \) be the quotient resulting from the exact division of \( P \) by \( v \), we get from (1)

\[ q = \frac{P}{v} = 100 - \frac{100 - p}{v} \]  \hspace{1cm} (2)

(2) guarantees that \( v \leq 100 \) since \( q \) cannot be an integer for \( v > 100 \).

By the conditions of the problem, \( q \) is also the last nonzero remainder appearing in the division process, so we have \( q \leq 9v \), or by (2)

\[ 9v^2 - 100v + 100 - p \geq 0. \]
Hence

\[ v \leq \frac{100 - \sqrt{6400 + 36p}}{18} \quad \text{or} \quad v \geq \frac{100 + \sqrt{6400 + 36p}}{18}. \]

Taking \( p = 0 \) provides the useful bounds \( v \leq 10/9 \) or \( v \geq 10 \).

We first take \( v \geq 10 \). Then by (2) \( q = 90 + k \) for some integer \( k \) such that \( 0 \leq k \leq 9 \). But since \( q \) is the last nonzero remainder and \( k \) is the units digit of \( q \), we must have \( k = q/v \). Hence \( 90 + k = q = kv \) or

\[ \frac{90}{v - 1} = k. \] (3)

Since \( 0 \leq k \leq 9 \), (3) yields \( (v, k) = (11, 9), (15, 6), (19, 5), (31, 3), (46, 2) \) and \( (91, 1) \). Substituting \( q = 90 + k \) into (2) and rearranging yields

\[ p = 100 - (10 - k)v. \] (4)

Using each of the above possibilities for \( (v, k) \), we discard the last three since they yield \( p < 0 \). Hence we have the following solutions:

- \( v = 11, \ k = 9, \ p = 89, \ P = 1089; \)
- \( v = 15, \ k = 6, \ p = 36, \ P = 1536; \)
- \( v = 19, \ k = 5, \ p = 5, \ P = 1805. \)

It remains to consider \( v \leq 10/9 \) or \( v = 1 \). Here we have nine trivial solutions \( (v, k, p) = (1, k, k) \) where \( 1 \leq k \leq 9 \). They correspond to the first nine problems published in *Eureka* (*Crux’s predecessor*).

Hence the next and last problem number satisfying the conditions of the problem will be *Crux* 1805 appearing in volume 19.

*Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; and K.R.S. SASTRY, Addis Ababa, Ethiopia. One incorrect solution was received.*

*Readers may like to contribute an appropriate problem to be numbered 1805, to carry on the “tradition” newly established by Janous! The editor suggests that such a problem (i) involve the number 1805 and possibly the volume number (19) or even the year (which will be 1993), and (ii) have exactly one integer solution larger than 1805. (Not too large!)*

\[ * \quad * \quad * \quad * \quad * \quad * \quad * \]

**1537.** [1990: 110] *Proposed by Isao Ashiba, Tokyo, Japan.*

\( ABC \) is a right triangle with right angle at \( A \). Construct the squares \( ABDE \) and \( ACFG \) exterior to \( \triangle ABC \), and let \( P \) and \( Q \) be the points of intersection of \( CD \) and \( AB \), and of \( BF \) and \( AC \), respectively. Show that \( AP = AQ \).
I. Solution by O. Johnson, student, King Edward’s School, Birmingham, England.

Let $AB = AE = p$ and $AC = AG = q$.

Then $\triangle CAP$ is similar to $\triangle CED$, so

$$\frac{AP}{q} = \frac{AP}{AC} = \frac{DE}{EC} = \frac{p}{p + q},$$

and therefore

$$AP = \frac{pq}{p + q}.$$

Also $\triangle BAQ$ is similar to $\triangle BGF$, so

$$\frac{AQ}{p} = \frac{AQ}{AB} = \frac{FG}{GB} = \frac{q}{p + q},$$

and therefore

$$AQ = \frac{pq}{p + q} = AP.$$

II. Comment by Hidetosi Fukagawa, Aichi, Japan.

Several interesting facts about the figure of this problem are shown in Shiko Iwata, Encyclopedia of Geometry Vol. 3 (1976), published in Japanese. They include, as well as this problem itself,

(i) $BP : CQ = (AB)^2 : (AC)^2$;

(ii) if $DE$ meets $GF$ in $M$, then $AM \perp BC$ and the three lines $AM, DC, BF$ meet in a point $H$;

(iii) $1/AH = 1/AL + 1/BC$, where $AM$ meets $BC$ in $L$;

(iv) $BD' = CF'$, where $D'$ and $F'$ are the feet of the perpendiculars from $D$ and $F$ to $BC$;

(v) $DD' + FF' = BC$.

[Fukagawa then gave proofs of (i)–(v) and Crux 1537. —Ed.]

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; J. CHONÉ, Lycée Blaise Pascal, Clermont-Ferrand, France; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; ANTONIO LUIZ SANTOS, Rio de Janeiro, Brazil; BEATRIZ MARGOLIS, Paris, France; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; H.J. MICHEL WIJERS, Eindhoven, The Netherlands; KA-PING YEE, student, St. John’s-Ravenscourt School, Winnipeg, Manitoba; and the proposer.
Heuver found the problem with solution on pp. 83–84 of Coxeter and Greitzer, Geometry Revisited, L.W. Singer, New York.

McCallum notes that another problem, rather familiar, has the same answer $pq/(p + q)$ as was obtained for $AP$ and $AQ$ above: namely, given two vertical posts of heights $p$ and $q$ on a flat plain, and wires connecting the top of each post with the bottom of the other, find the height of the intersection of the wires. He wonders if there is a nice way to turn each problem into the other.

* * * * *

Find all functions $y = f(x)$ with the property that the line through any two points $(p, f(p)), (q, f(q))$ on the curve intersects the $y$-axis at the point $(0, -pq)$.

Solution by Ka-Ping Yee, student, St. John’s-Ravenscourt School, Winnipeg, Manitoba.
All functions $y = f(x) = x(c + x)$, where $c$ is some constant, have this property, and no other functions do. We can show this as follows.

If a line passes through the points $(p, f(p))$ and $(q, f(q))$, then

$$f(p) = mp + b, \quad f(q) = mq + b.$$  

Eliminating the slope $m$ and solving for the $y$-intercept $b$ we get

$$b = \frac{pf(q) - qf(p)}{p - q} = -pq \quad \text{(as given)},$$

which implies

$$pf(q) - qf(p) = pq(q - p)$$

and so

$$\frac{f(q)}{q} - q = \frac{f(p)}{p} - p.$$  

Since this is true for all [nonzero] real $p$ and $q$, $f(x)/x - x$ has a constant value for all $x \neq 0$. If we let this constant value be $c$, then $f(x) = x(c + x)$.

[Editor’s note. To this solution should be added two points, missed by many solvers: (i) from the statement of the problem, $f(0)$ must be $0$, and the functions $f(x) = x(c + x)$ all have this property, thus $f(x) = x(c + x)$ for all $x$ (not just $x \neq 0$); (ii) all functions $f(x) = x(c + x)$ do in fact satisfy the problem.]  

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; J. CHONÉ, Lycée Blaise Pascal, Clermont-Ferrand, France; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; STEPHEN D HNIDEI and ROBERT PIDGEON, students, University of British Columbia; WALTHER JANOUS, Ursulinkgymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland;
The problem is the converse of a result proved in Herb Holden’s article “Chords of the parabola”, Two Year College Math. Journal, June 1982, pp. 186–190, and was suggested by that article.

* * * * *


If $\alpha, \beta, \gamma$ are the angles, $s$ the semiperimeter, $R$ the circumradius and $r$ the inradius of a triangle, prove or disprove that

$$\sum \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2} \leq \left( \frac{2R - r}{s} \right)^2,$$

where the sum is cyclic.

Solution by Stephen D. Hnidei, student, University of British Columbia.

Starting with

$$\tan \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan \frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}, \quad (1)$$

where $a, b, c$ are the sides opposite the angles $\alpha, \beta, \gamma$ respectively, the given inequality becomes

$$(s-a)^2 + (s-b)^2 + (s-c)^2 \leq (2R - r)^2,$$

or equivalently

$$a^2 + b^2 + c^2 \leq (2R - r)^2 + s^2. \quad (2)$$

Using the fact that

$$\frac{a^2 + b^2 + c^2}{2} + r(4R + r) = s^2, \quad (3)$$

(2) becomes

$$a^2 + b^2 + c^2 \leq 8R^2 + 4r^2.$$

This is inequality 5.14, page 53 of Bottema et al, Geometric Inequalities.

[Editor’s note. (1) can be proved from the formulas

$$\tan \frac{\alpha}{2} = \frac{r}{s-a}, \quad \text{etc. and} \quad r^2 s = (s-a)(s-b)(s-c).$$

(3) is proved on page 52 of Mitrović et al, Recent Advances in Geometric Inequalities.]

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; JACK GARFUNKEL, Flushing, N.Y.; WALTHER JANOUS, Ursulinenrinkasium, Innsbruck, Austria;
MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Penn State University at Harrisburg; and BOB PRIELIPP, University of Wisconsin-Oshkosh.

Most solvers reduced the given inequality to item 5.8, page 50 of Bottema et al, Geometric Inequalities. Thus this problem and Crux 1534 (solution this issue) are equivalent!

* * * * *


For $k$ a positive odd integer, define a sequence $<a_n>$ by: $a_0 = 1$ and, for $n > 0$,

$$a_n = \begin{cases} a_{n-1} + k & \text{if } a_{n-1} \text{ is odd}, \\ a_{n-1}/2 & \text{if } a_{n-1} \text{ is even}, \end{cases}$$

and let $f(k)$ be the smallest $n > 0$ such that $a_n = 1$. Find all $k$ such that $f(k) = k$.

Solution by Emilio Fernández Moral, I.B. Sagasta, Logroño, Spain.

We will see that $f(k) = k$ if and only if $k = 3^q$ with $q \geq 1$.

First we look at some particular cases. For $k = 15$, the sequence is:

$$<a_n>: 1, 16, 8, 4, 2, 1, \cdots$$

and $f(15) = 5$.

For $k = 13$, we get

$$<a_n>: 1, 14, 7, 20, 10, 5, 18, 9, 22, 11, 24, 12, 6, 3, 16, 8, 4, 2, 1, \cdots$$

and $f(13) = 18$.

For $k = 9$, we get

$$<a_n>: 1, 10, 5, 14, 7, 16, 8, 4, 2, 1, \cdots$$

and $f(9) = 9$.

We can work better with the “auxiliary” sequence $<t_n>$ defined by: $t_0 = 1$, and for $n > 0$

$$t_n = \begin{cases} (t_{n-1} + k)/2 & \text{if } t_{n-1} \text{ is odd,} \\ t_{n-1}/2 & \text{if } t_{n-1} \text{ is even;} \end{cases}$$

and let $g(k)$ be also the smallest $n > 0$ such that $t_n = 1$. For the same values of $k$, we have

$$\begin{align*}
(k = 15) & \quad <t_n>: 1, 8, 4, 2, 1, \cdots \quad \text{and } g(15) = 4; \\
(k = 13) & \quad <t_n>: 1, 7, 10, 5, 9, 11, 12, 6, 3, 8, 4, 2, 1, \cdots \quad \text{and } g(13) = 12; \\
(k = 9) & \quad <t_n>: 1, 5, 7, 8, 4, 2, 1, \cdots \quad \text{and } g(9) = 6.
\end{align*}$$

Evidently, each odd $t_i (i \geq 0)$ shortens by one term the length of sequence $<a_n>$ in comparison to sequence $<t_n>$, so:

$$\begin{align*}
f(15) &= g(15) + 1 = g(15) + \text{card}\{1\}, \\
f(13) &= g(13) + 6 = g(13) + \text{card}\{1, 7, 5, 9, 11, 3\}, \\
f(9) &= g(9) + 3 = g(9) + \text{card}\{1, 5, 7\}.
\end{align*}$$
In general, once we prove that \( g(k) \) is finite for every positive odd integer \( k \), then we shall have that
\[
f(k) = g(k) + \text{card}\{ \text{odd } t_i : 0 \leq i \leq g(k) \}.
\] (1)

We shall show that the coincidence between the sequence \( \langle t_n \rangle \) (reversed) and the sequence of residues of the \( 2^n \)'s modulo \( k \), for example,
\[
\begin{align*}
&< 2^n \text{ modulo } 15 > : 1, 2, 4, 8, 1, \cdots , \\
&< 2^n \text{ modulo } 13 > : 1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, \cdots , \\
&< 2^n \text{ modulo } 9 > : 1, 2, 4, 8, 7, 5, 1, \cdots 
\end{align*}
\]
is not accidental. In fact we can prove that, for every odd \( k > 1 \) (already excluding the case \( k = 1 \), for which \( f(1) = 2 \)), we have:

(a) \( t_n < k \) for all \( n; \)

(b) \( 2^n t_n \equiv 1 \mod k \) for all \( n; \) and so

(c) if \( h \) is a positive integer such that \( 2^h \equiv 1 \mod k \), then \( t_n \equiv 2^{h-n} \mod k \) for \( n \leq h \), and so the period of the sequence \( \langle t_n \rangle \) is
\[
g(k) = \min \{h > 0 : 2^h \equiv 1 \mod k \}(= \text{ the order of } 2 \mod k).
\]

Proof of (a). By induction. Note that
\[
t_0 = 1 < k , \quad t_1 = \frac{1+k}{2} < k.
\]
If we suppose \( t_{n-1} < k \) and we put \( t_{n-1} \equiv \varepsilon \mod 2 \) (\( \varepsilon = 0 \) or \( 1 \)), then
\[
t_n = \frac{t_{n-1}}{2} + \varepsilon \cdot \frac{k}{2} < \frac{k}{2} + \frac{k}{2} = k.
\]

Proof of (b). By induction. Note that \( t_0 = 1 \) and \( 2t_1 = 1 + k \equiv 1 \mod k \). If we suppose that \( 2^{n-1} t_{n-1} \equiv 1 \mod k \), then we have (as above)
\[
2^n t_n = 2^n \left( \frac{t_{n-1}}{2} + \varepsilon \cdot \frac{k}{2} \right) = 2^{n-1} t_{n-1} + k 2^{n-1} \varepsilon
\]
\[
\equiv 2^{n-1} t_{n-1} \mod k \equiv 1 \mod k.
\]

Proof of (c). According to (b), \( 2^n t_n \equiv 1 \mod k \), and (as \( k \) is odd) by division \( t_n \equiv 2^{h-n} \mod k \) for every \( n \leq h \). By this and (a), the terms \( t_1, t_2, \ldots, t_h \) of the sequence \( \langle t_n \rangle \) are, in opposite order, just the same as the terms \( 2^0 \mod k, 2^1 \mod k, \ldots, 2^{h-1} \mod k \) of the sequence \( \langle 2^n \mod k \rangle \), for every positive \( h \) such that \( 2^h \equiv 1 \mod k \). The periods of \( \langle t_n \rangle \) and \( \langle 2^n \mod k \rangle \) are therefore equal.

Remembering (1) above, we have now that sequence \( \langle a_n \rangle \) is pure periodic for all odd \( k \), with period \( f(k) = \ell + s \) with
\[
\ell = \text{the order of } 2 \mod k,
\]
\[
s = \text{card} \{\text{odd residues of } 2^i \mod k , 0 < i \leq \ell \}.
\]
Finally we answer the problem, that is, we prove  
\[ f(k) = k \text{ if and only if } k = 3^q \text{ with } q \geq 1. \]

\(\implies\) It is known that \(\ell\) divides the Euler phi-function \(\varphi(k)\), because \(2^{\varphi(k)} \equiv 1 \) mod \(k\) (\(k\) and 2 are relatively prime) and \(\ell\) is the smallest exponent which satisfies the congruence. On the other hand, evidently \(s < \ell\). If \(\ell \neq \varphi(k)\) then \(\ell \leq \varphi(k)/2 < k/2\) and so

\[ f(k) = \ell + s < \frac{k}{2} + \frac{k}{2} = k, \]

a contradiction. Thus \(\ell = \varphi(k)\). In this case the sequence \(< 2^i \text{ mod } k : 1 \leq i \leq \ell >\) contains all the positive numbers less than \(k\) and prime to \(k\). If \(m < k\) is prime to \(k\), then \(k - m\) is prime to \(k\) too and has opposite parity to \(m\) since \(k\) is odd; therefore

\[ s = \frac{\varphi(k)}{2} = \text{card}\{\text{all positive odd integers } < k \text{ and prime to } k\}. \]

So we have by the hypothesis

\[ k = f(k) = \ell + s = \varphi(k) + \frac{\varphi(k)}{2} = \frac{3}{2}\varphi(k), \]

or

\[ \frac{2k}{3} = \varphi(k) = k \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots, \]

where \(p_1, p_2, \ldots\) are the distinct prime factors of \(k\). Thus

\[ \frac{2}{3} = \prod_{p \text{ prime}} \frac{p-1}{p} \]

and consequently the only prime factor of \(k\) can be \(p = 3\). Therefore \(k = 3^q\) with \(q \geq 1\).

\(\iff\) There remains to prove that

\(\ast\) for every \(k = 3^q\) with \(q \geq 1\), the order of 2 modulo \(k\) is \(\ell = \varphi(k) = 2 \cdot 3^{q-1}\).

It then follows that \(s = \varphi(k)/2 = 3^{q-1}\) and so

\[ f(k) = \ell + s = 2 \cdot 3^{q-1} + 3^{q-1} = 3^q = k, \]

as claimed.

The proof of \(\ast\) is by induction on \(q\). For \(k = 3\), \(\varphi(k) = 2\), and we have \(2^2 \equiv 1 \) mod 3 but \(2^1 \not\equiv 1 \) mod 3. Moreover \(2^2 = 1 + 1 \cdot 3\) where the second 1 is prime to 3. Suppose now that \(2 \cdot 3^{q-1}\) is the order of 2 modulo \(3^q\), and also that

\[ 2^{2 \cdot 3^{q-1}} = 1 + d \cdot 3^q \]
where \( d \) is prime to 3. Then we have
\[
2^{2 \cdot 3^q} = (1 + d \cdot 3^q)^3 = 1 + 3 \cdot 3^q d + 3 \cdot 3^{2q} d^2 + 3^{3q} d^3
= 1 + 3^{q+1} (d + 3c) \quad \text{for some integer } c
= 1 + 3^{q+1} d',
\]
where \( d' \) is prime to 3, since \( d \) is. Thus \( 2^{2 \cdot 3^q} \equiv 1 \mod 3^{q+1} \) (which of course follows by Euler’s theorem) and moreover, if \( 2^h \equiv 1 \mod 3^{q+1} \) with \( h < 2 \cdot 3^q \), then in particular we will have \( 2^h \equiv 1 \mod 3^q \), so \( h \) must be a multiple of \( 2 \cdot 3^{q-1} \), that is, \( h = 2 \cdot 3^{q-1} \) or \( h = 2^2 \cdot 3^{q-1} \). But
\[
2^{2 \cdot 3^{q-1}} = 1 + d \cdot 3^q \not\equiv 1 \mod 3^{q+1}
\]
since \( d \) is prime to 3, and (as \( 2^2 = 2 \cdot 2 \))
\[
2^{2^2 \cdot 3^{q-1}} = (2^{2 \cdot 3^{q-1}})^2 = (1 + 3^q d)^2
= 1 + 2 \cdot 3^q d + 3^2 q d^2 \equiv 1 + 2 \cdot 3^q d \mod 3^{q+1}
\not\equiv 1 \mod 3^{q+1}
\]
since \( d \) is prime to 3. Thus the order of \( 2 \) modulo \( 3^{q+1} \) is \( 2 \cdot 3^q \), and (*) is proved.

Also solved (the same way) by C. FESTRAETS-HAMOIR, Brussels, Belgium; MARCIN E. KUCZMA, Warszawa, Poland; and the proposers. Partial solutions were sent in by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Barnberg, Germany; and by P. PENNING, Delft, The Netherlands (they each got one of the directions of the above “if and only if” statement). Two other readers submitted the correct answer with no proof.

As Engelhaupt points out, the result (*) in the above proof is known. For example, see Theorem 8.10, page 261 of K.H. Rosen, Elementary Number Theory and its Applications (2nd edition), Addison Wesley, 1988.

* * * * *


I₁ is the excenter of \( \Delta A₁A₂A₃ \) corresponding to side \( A₂A₃ \). \( P \) is a point in the plane of the triangle, and \( A₁P \) intersects \( A₂A₃ \) in \( P₁ \). \( I₂, I₃ \), \( P₂, I₃ \), \( P₃, I₃ \) are analogously defined. Prove that the lines \( I₁P₁, I₂P₂, I₃P₃ \) are concurrent.

Solution by R.H. Eddy, Memorial University of Newfoundland.

More generally, let \( \Delta Q₁Q₂Q₃ \) be circumscribed about \( \Delta A₁A₂A₃ \) such that \( A₁Q₁ \cap A₂Q₂ \cap A₃Q₃ = Q(q₁, q₂, q₃) \); then the lines \( P₁Q₁, P₂Q₂, P₃Q₃ \) are concurrent.

We use trilinear coordinates taken with respect to \( \Delta A₁A₂A₃ \). If \( P \) has coordinates \((p₁, p₂, p₃)\) then those for \( P₁, P₂, P₃ \) are \((0, p₂, p₃), (p₁, 0, p₃), (p₁, p₂, 0)\) respectively. Since the coordinates for \( Q₁, Q₂, Q₃ \) are \((-q₁, q₂, q₃), (q₁, -q₂, q₃), (q₁, q₂, -q₃)\), the line coordinates of \( P₁Q₁, P₂Q₂, P₃Q₃ \) are given respectively by the rows of the determinant
\[
\begin{vmatrix}
| p₂q₁ - p₃q₂ - p₃q₁ | & p₂q₁ & -q₁p₂ & -q₁p₃ \\
| p₃q₂ & p₃q₁ - p₁q₃ & -p₁q₂ | & p₃q₁ & -q₂p₃ & -q₂p₁ \\
| -p₂q₃ & p₁q₃ & p₁q₂ - p₂q₁ | & -p₂q₃ & p₁q₃ & p₁q₂ - p₂q₁ \\
\end{vmatrix}
\]
which is seen, by adding the rows together, to have the value zero. Thus the lines are concurrent.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; HIDETOSI FUKAGAWA, Aichi, Japan; P. PENNING, Delft, The Netherlands; JOHN RAUSEN, New York; D.J. SMEENK, Zaltbommel, The Netherlands; D. SOKOLOWSKY, Williamsburg, Virginia; E. SZEKERES, Turramurra, Australia; and the proposer.

The generalization given by Eddy was also found by Rausen, who later located it in Aubert et Papelier, Exercices de Géométrie Analytique, Vol. 1, 10th ed., Paris, 1957, problem 52, p. 35. Rausen also points out that the lines $I_1P_1$, $I_2P_2$, $I_3P_3$ may be parallel rather than concurrent: just pick $P_1$ on line $A_2A_3$ and $P_2$ on $A_3A_1$ such that lines $I_1P_1$ and $I_2P_2$ are parallel, then define $P$ to be $A_1P_1 \cap A_2P_2$.

* * * * *


Show that the circumradius of a triangle is at least four times the inradius of the pedal triangle of any interior point.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $R_1, R_2, R_3$ and $r_1, r_2, r_3$ be the distances of the interior point $P$ from the vertices and sides, respectively, of triangle $A_1A_2A_3$ having sides $a_1, a_2, a_3$. The sides of the pedal triangle are given by

$$R_i \sin A_i = \frac{a_i R_i}{2R}, \quad i = 1, 2, 3$$

(e.g. [1], p. 296), and its area is given by

$$F_{\text{ped}} = \frac{1}{2} \sum r_2 r_3 \sin A_i = \frac{1}{4R} \sum a_1 r_2 r_3$$

where the sums are cyclic. (This can be seen by the trigonometric area formula and the fact that $r_2$ and $r_3$ make the angle $\pi - A_1$, etc.) Therefore the claimed inequality $R \geq 4r_{\text{ped}}$ reads $Rs_{\text{ped}} \geq 4F_{\text{ped}}$ or

$$R \sum a_1 R_1 \geq 4 \sum a_1 r_2 r_3. \quad (1)$$

Now it is known ([2], item 12.19) that

$$\sum a_1 R_1 \geq 2 \sum a_1 r_1 = 4F.$$

Therefore (1) would follow from $RF \geq \sum a_1 r_2 r_3$, i.e.,

$$a_1 a_2 a_3 \geq 4 \sum a_1 r_2 r_3.$$

But this inequality can either be found in [1] (p. 333, item 7.1 or p. 339, item 7.28) or in Crux ([1990: 64] or [1987: 260]).
One root of $x^3 + ax + b = 0$ is $\lambda$ times the difference of the other two roots ($|\lambda| \neq 1$). Find this root as a simple rational function of $a$, $b$ and $\lambda$.

Solution by Jean-Marie Monier, Lyon, France.

Letting the roots be $x_1, x_2, x_3$, we have

$$x_1^2 = [\lambda(x_2 - x_3)]^2 = \lambda^2[(x_2 + x_3)^2 - 4x_2x_3] = \lambda^2\left(x_1^2 + \frac{4b}{x_1}\right)$$

(since $x_1 + x_2 + x_3 = 0$ and $x_1x_2x_3 = -b$), hence $(1 - \lambda^2)x_1^3 = 4\lambda^2b$ or

$$x_1^3 = \frac{4\lambda^2b}{1 - \lambda^2}.$$

Thus

$$ax_1 + b = -x_1^3 = \frac{4\lambda^2b}{\lambda^2 - 1},$$

so

$$x_1 = \frac{1}{a}\left(\frac{4\lambda^2b}{\lambda^2 - 1} - b\right) = \frac{b(3\lambda^2 + 1)}{a(\lambda^2 - 1)}.$$

[Editor’s note. Monier’s solution actually contained a small error, but in other respects was the nicest one received, so the error has been corrected in the above write-up.]

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MATHEW ENGLANDER, student, University of Waterloo; M. MERCEDES SÁNCHEZ BENITO, Madrid, and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; STEPHEN D. HNIDEI, student, University of British Columbia; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; O. JOHNSON, student, King Edward’s School, Birmingham, England; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; BEATRIZ MARGOLIS, Paris, France; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

As was pointed out by several solvers, the solution is valid only for $a \neq 0$. 

References:

Also solved by JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; MURRAY S. KLAMKIN, University of Alberta; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

* * * * *
A sphere is said to be inscribed in the skeleton of a convex polyhedron if it is tangent to all the edges of the polyhedron. Given a convex polyhedron $P$ and a point $O$ inside it, suppose a sphere can be inscribed into the skeleton of each pyramid spanned by $O$ and a face of $P$.

(a) Prove that if every vertex of $P$ is the endpoint of exactly three edges then there exists a sphere inscribed into the skeleton of $P$.

(b) Is this true without the assumption stated in (a)?

**Solution by the proposer:**

(a) Label the faces $F_1, \ldots, F_n$ in such a way that $F_1$ and $F_2$ be adjacent and every successive $F_k$ ($3 \leq k \leq n$) be adjacent to some two previous faces. [For instance, $F_3$ would be the third face meeting at one of the endpoints of the common edge of $F_1$ and $F_2$.—Ed.] Let $P_j$ be the pyramid of base $F_j$ and vertex $O$, let $D_j$ be the disc inscribed in $F_j$ (it does exist) and let $\ell_j$ be the line through the center of $D_j$ perpendicular to the plane of $F_j$. The two spheres inscribed into the skeletons of $P_1$ and $P_2$ touch the common edge of $F_1$ and $F_2$ in the same point (they both contain the incircle of triangle $P_1 \cap P_2$). The two planes passing through that point and through line $\ell_1$, resp. $\ell_2$, are perpendicular to the edge, hence they are identical. Thus $\ell_1$ and $\ell_2$ intersect.

Face $F_3$ is adjacent to $F_1$ and $F_2$. The same argument shows that $\ell_3$ cuts $\ell_1$ and $\ell_2$. Since $\ell_1, \ell_2, \ell_3$ do not lie in the same plane, we infer that they have exactly one point in common.

By induction, each successive line $\ell_4, \ldots, \ell_n$ passes through this point, which is therefore equally distant from all edges of $P$ [because it is equidistant from all the edges of any one face]; the claim results.

(b) The assumption given in (a) is indeed essential. As an example consider the following construction. $ABC$ is an equilateral triangle of side $a$, $O$ is its center. Points $D$ and $E$ lie symmetrically with respect to plane $ABC$, at a distance $h$ from it; $O$ is the midpoint of $DE$. Polyhedron $P$ is defined as the union of pyramids $ABCD$ and $ABCE$. A sphere can be inscribed into the skeleton of any one of the six pyramids spanned by $O$ and the faces of $P$ if and only if $h = (\sqrt{3} - 1)a/4$ (verification is routine and therefore omitted). A sphere inscribed into the skeleton of $P$ exists if and only if $h = a/3$.

There was one incorrect solution sent in, which failed to take into account the condition in part (a).
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CONTENTS

On an Idea of Groenman .............................. R.H. Eddy 193

The Olympiad Corner: No. 127 ...................... R.E. Woodrow 195

Problems: 1661–1670 ...................................... 206

Solutions: 1542, 1546–1552 ................................... 208
ON AN IDEA OF GROENMAN

R.H. Eddy

In *Crux* [1987: 256] and [1987: 321], the late J.T. Groenman, a prolific and much appreciated contributor, proposed the following two related problems.

**1272.** (solution [1988: 256]) Let $A_1A_2A_3$ be a triangle. Let the incircle have center $I$ and radius $r$, and meet the sides of the triangle at points $P_1, P_2, P_3$. Let $I_1, I_2, I_3$ be the excenters and $\rho_1, \rho_2, \rho_3$ the exradii. Prove that
(a) the lines $I_1P_1, I_2P_2, I_3P_3$ concur at a point $S$;
(b) the distances $d_1, d_2, d_3$ of $S$ to the sides of the triangle satisfy

$$
d_1 : d_2 : d_3 = \rho_1 : \rho_2 : \rho_3.
$$

**1295.** (solution [1989: 17-19]) Let $A_1A_2A_3$ be a triangle with $I_1, I_2, I_3$ the excenters and $B_1, B_2, B_3$ the feet of the altitudes. Show that the lines $I_1B_1, I_2B_2, I_3B_3$ concur at a point colinear with the incenter and circumcenter of the triangle.

A third point, with a similar construction, may be found in Nagel [9], where in Groenman’s notation we may write

Let $S_1, S_2, S_3$ denote respectively the midpoints of the sides $A_2A_3, A_3A_1, A_1A_2$ of the given triangle; then the lines $I_1S_1, I_2S_2, I_3S_3$ concur at the MITTENPUNKT of the given triangle.

We found out about this interesting point, while on sabbatical leave in Europe in the fall of 1986, from Peter Baptist, a faculty member at Bayreuth University who has done considerable work in the geometry of the triangle, particularly as it relates to special points. (For an example of his work, see [3].) A direct translation would seem to yield the term *middlespoint* in English, and this is a good description since the point is constructed using middles, i.e., centres of circles and midpoints of line segments. This term is used in [5] and [6], but since it is rather clumsy we shall use *mittenpunkt* throughout this note.

Let us consider the point from a different angle. Consider the pair of triangles $S_1S_2S_3$ and $I_1I_2I_3$ which are in perspective with the triangle $A_1A_2A_3$ from the centroid and incentre respectively. Then the triangles $S_1S_2S_3$ and $I_1I_2I_3$ are in perspective from the mittenpunkt of $A_1A_2A_3$. While Nagel’s proof is synthetic—in fact he seems to dislike other types, especially trigonometric—the problem is easily solved analytically using trilinear coordinates. Since these have appeared in *Crux* several times, further details will not be given here. It suffices to remark that the trilinear coordinates of the mittenpunkt are $(s-a_1, s-a_2, s-a_3)$ where, as usual, $s$ is the semiperimeter of the given triangle. Since the construction seems to be something of an unnatural “buddying-up”, so to speak, of the incentre and the centroid, one immediately gets the feeling that some sort of generalization is lurking around. Further justification for such a possibility is immediately obtained by replacing the centroid by the symmedian point, see *Crux* [11], which has the rather nice...
coordinates \((\sin \alpha, \sin \beta, \sin \gamma) = (a_1, a_2, a_3)\). It is an elementary exercise to show that the corresponding lines are again concurrent.

The following generalization is given in [6]. A point \(P\) in the interior of the given triangle determines an inscribed triangle \(P_1P_2P_3\), where \(P_i = A_iP \cap A_{i+1}A_{i+2}, i = 1, 2, 3.\) (Subscripts here and below are taken modulo 3.) A second interior point \(Q\) determines a circumscribed triangle in the following manner. Consider the harmonic conjugates \(Q'_1, Q'_2, Q'_3\) of \(Q_1, Q_2, Q_3\) with respect to the point pairs \((A_2, A_3), (A_3, A_1),\) and \((A_1, A_2)\) which lie on a line \(q\), the trilinear polar of \(Q(y_i)\) with respect to the given triangle [1]. Let \(Q^i = A_{i+1}Q'_{i+1} \cap A_{i+2}Q'_{i+2}\). Then the trilinear coordinates of \(Q^i\) are \((-1)^{\delta_{ij}} y_j\), where \(\delta_{ij} = 1\) if \(i = j\) and 0 otherwise. The following theorem follows readily.

The triangles \(P_1P_2P_3\) and \(Q^1Q^2Q^3\) are perspective from the point \(T = \cap P_iQ^i, i = 1, 2, 3.\)

The restriction to \(P\) being in the interior of the triangle is unnecessary: see the solution of Crux 1541 [1991: 189]. For alternative formulations, see [1] or [2]. We are told that the theorem also appears in [8] and [10]. (We thank the referees for supplying references [1], [2], [8] and [10] which were previously unknown to us.)

For Groenman’s problem 1272, \(P\) is the Gergonne point and \(Q\) is the incentre, while for 1295, \(P\) is the orthocentre and \(Q\) is again the incentre. Also noted in [6] is the fact that the point in 1272 is the isogonal conjugate of the mittenpunkt. This problem was discovered at the proofreading stage of [6] and thus was able to be included.

It is interesting that Groenman seemed to be approaching the same generalization. In his solution to 1295 he has \(P\) in general position while \(Q\) is still the incentre. A related class of points referred to by Nagel as interior mittenpunkts (defined by replacing one of the excentres by the incentre and interchanging the other two) is also given in [9]. This class is also generalized in [6]. A dual notion for \(T\), the mittenlinie (middle line), is given in [5].

For those readers who have access to a symbolic manipulation program, the following is an elementary exercise.

*Under what circumstances are \(P, Q, \) and \(T\) collinear?*

If the trilinear coordinates of \(P\) and \(Q\) are \((z_i)\) and \((y_i)\) respectively, \(i = 1, 2, 3,\) then it is not too difficult to show that the coordinates of \(T(x_i)\) are

\[
(y_1 \left( -\frac{y_1}{z_1} + \frac{y_2}{z_2} + \frac{y_3}{z_3} \right), y_2 \left( \frac{y_1}{z_1} - \frac{y_2}{z_2} + \frac{y_3}{z_3} \right), y_3 \left( \frac{y_1}{z_1} + \frac{y_2}{z_2} - \frac{y_3}{z_3} \right)).
\]

The collinearity condition then leads to the equation

\[
(y_1 z_2 - y_2 z_1)(y_2 z_3 - y_3 z_2)(y_3 z_1 - y_1 z_3) = 0.
\]

Hence \(y_i/y_{i+1} = z_i/z_{i+1}\) for some \(i,\) and thus \(P, Q, T\) must lie on a line through a vertex of the given triangle.

An exhaustive search for some other references to this point has met with only one partial success. In Gallatly [7], it is mentioned only as the Lemoine point \(\Lambda\) of the triangle \(I_1I_2I_3\). The coordinates of \(\Lambda\) are given as \((s - a_1, s - a_2, s - a_3,\) thus it is Nagel’s
mittenpunkt but under another guise. I am surprised that a point with such a pretty and ready construction seems not to have found its way into the modern geometry of the triangle. Perhaps the reader can shed some new light on the problem.

References:

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* * * * *

THE OLYMPIAD CORNER

No. 127

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

In recent years the September number of the Corner has been the issue in which the year’s IMO is discussed. I rely on our readers and my colleagues who may attend the IMO for information, and there is frequently a panic and some delay in getting the September number out since mathematicians are notorious for forgetting to carry out routine tasks like sending off copies of questions and so on. We have also been experiencing some delays with the switch over to LaTeX and so we have decided to hold over to a later issue the 1991 IMO results. We begin this number with problems submitted to the jury but not used for
the 31st IMO in China. I am indebted to Professor Andy Liu of the University of Alberta, who was involved with training the Chinese team, for taking the time to send me these problems.

1. Proposed by Australia.
   The integer 9 can be written as a sum of two consecutive integers: $9 = 4 + 5$; moreover, it can be written as a sum of more than one consecutive integer in exactly two ways, namely $9 = 4 + 5 = 2 + 3 + 4$. Is there an integer which can be written as a sum of 1990 consecutive integers and which can be written as a sum of more than one consecutive integer in exactly 1990 ways?

2. Proposed by Canada.
   Given $n$ countries with 3 representatives each, a list of $m$ committees $A(1), A(2), \ldots, A(m)$ is called a cycle if
   (1) each committee has $n$ members, one from each country;
   (2) no two committees have the same membership;
   (3) for $1 \leq i \leq m$, committee $A(i)$ and committee $A(i + 1)$ have no member in common, where $A(m + 1)$ denotes $A(1)$;
   (4) if $1 < |i - j| < m - 1$, then committees $A(i)$ and $A(j)$ have at least one member in common.
   Is it possible to have a cycle of 1990 committees with 11 countries?

3. Proposed by Czechoslovakia.
   Assume that the set of all positive integers is decomposed into $r$ disjoint subsets $N = A_1 \cup \cdots \cup A_r$. Prove that one of them, say $A_i$, has the following property: there exists a positive integer $m$ such that for any $k$, one can find numbers $a_1, a_2, \ldots, a_k$ in $A_i$ with $0 < a_{j+1} - a_j \leq m$, $1 \leq j \leq k - 1$.

4. Proposed by France.
   Given $\triangle ABC$ with no side equal to another side, let $G$, $K$ and $H$ be its centroid, incentre and orthocentre, respectively. Prove that $\angle GKH > 90^\circ$.

5. Proposed by Greece.
   Let $f(0) = f(1) = 0$ and
   $$f(n + 2) = 4^{n+2} f(n + 1) - 16^{n+1} f(n) + n 2^n,$$
   $n = 0, 1, 2, \ldots$. Show that the numbers $f(1989)$, $f(1990)$ and $f(1991)$ are divisible by 13.

6. Proposed by Hungary.
   For a given positive integer $k$, denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.

7. Proposed by Iceland.
   A plane cuts a right circular cone into two parts. The plane is tangent to the circumference of the base of the cone and passes through the midpoint of the altitude. Find the ratio of the volume of the smaller part to the volume of the whole cone.
8. Proposed by Ireland.
Let \(ABC\) be a triangle and \(\ell\) the line through \(C\) parallel to the side \(AB\). Let the internal bisector of the angle at \(A\) meet the side \(BC\) at \(D\) and the line \(\ell\) at \(E\). Let the internal bisector of the angle at \(B\) meet the side \(AC\) at \(F\) and the line \(\ell\) at \(G\). If \(GF = DE\) prove that \(AC = BC\).

On the coordinate plane a rectangle with vertices \((0, 0)\), \((m, 0)\), \((0, n)\) and \((m, n)\) is given where both \(m\) and \(n\) are odd integers. The rectangle is partitioned into triangles in such a way that

(1) each triangle in the partition has at least one side, to be called a “good” side, which lies on a line of the form \(x = j\) or \(y = k\) where \(j\) and \(k\) are integers, and the altitude on this side has length 1;

(2) each “bad” side is a common side of two triangles in the partition.
Prove that there exist at least two triangles in the partition each of which has two “good” sides.

Determine for which positive integers \(k\) the set \(X = \{1990, 1991, 1992, \ldots, 1990+k\}\) can be partitioned into two disjoint subsets \(A\) and \(B\) such that the sum of the elements of \(A\) is equal to the sum of the elements of \(B\).

11. Proposed by the Netherlands.
Unit cubes are made into beads by drilling a hole through them along a diagonal. The beads are put on a string in such a way that they can move freely in space under the restriction that the vertices of two neighbouring cubes are touching. Let \(A\) be the beginning vertex and \(B\) the end vertex. Let there be \(p \times q \times r\) cubes on the string where \(p, q, r \geq 1\).

(a) Determine for which values of \(p\), \(q\) and \(r\) it is possible to build a block with dimensions \(p\), \(q\) and \(r\). Give reasons for your answer.

(b) The same as (a) with the extra condition that \(A = B\).

* * *

The Canadian Mathematics Olympiad for 1991 saw two students tie for first prize. One of them went on to place in the top eight for the USAMO. The top prize winners are listed below:

First Prize \(\{\text{Ian Goldberg, J.P. Grossman}\}
Second Prize \(\text{Janos Csirik}
Third Prize \(\text{Adam Logan}
Fourth Prize \(\{\text{Jie Lou, Kevin Kwan, Peter Milley, Mark Van Raamsdonk}\}

* * *
The “official” solutions to the 1991 CMO are reproduced below with the permission of the CMO Committee of the Canadian Mathematical Society.

1991 CANADIAN MATHEMATICS OLYMPIAD
April 1991
Time: 3 hours

1. Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers $x, y, z$ for which $xyz \neq 0$.

   \textit{Solution.} Two solutions found by inspection are $(x, y, z) = (3, -1, 2)$ and $(10, 3, 7)$. Suppose a solution $(x, y, z) = (u, v, w)$ is given. Then for any positive integer $k$, $(x, y, z) = (k^{15}u, k^6v, k^{10}w)$ is also a solution.

2. Let $n$ be a fixed positive integer. Find the sum of all positive integers with the following property: in base 2, it has exactly $2n$ digits consisting of $n$ 1’s and $n$ 0’s. (The first digit cannot be 0.)

   \textit{Solution.} When $n = 1$, the sum is clearly 2. Let $n \geq 2$. The left digit is 1 and there are $\binom{2n-1}{n-1}$ possibilities for arranging $(n-1)$ 1’s and $n$ 0’s in the other $2n-1$ digital positions. Consider any digital position but the first. The digit in it is 1 for $\binom{2n-2}{n}$ of the numbers and 0 for $\binom{2n-2}{n-1}$ of the numbers, so that the sum of the digits in this position is $\binom{2n-2}{n}$. Hence the sum of all the numbers is

   $$\binom{2n-2}{n} (1 + 2 + \cdots + 2^{2n-2}) + \binom{2n-1}{n} 2^{2n-1} = \binom{2n-2}{n} (2^{2n-1} - 1) + \binom{2n-1}{n} 2^{2n-1}.$$  

3. Let $C$ be a circle and $P$ a given point in the plane. Each line through $P$ which intersects $C$ determines a chord of $C$. Show that the midpoints of these chords lie on a circle.

   \textit{Solution.} Let $O$ be the centre of circle $C$ and $X$ be the midpoint of a chord through $P$. Then $OX$ is perpendicular to $XP$, so that $X$ lies on that portion of the circle with diameter $OP$ which lies within $C$. (When $P = O$, the locus degenerates to the single point $O$.)

4. Ten distinct numbers from the set \{0, 1, 2, \ldots, 13, 14\} are to be chosen to fill in the ten circles in the diagram. The absolute values of the differences of the two numbers joined by each segment must be different from the values for all other segments. Is it possible to do this? Justify your answer.

   \textit{Solution.} Observe that each circle has an even number of edges emanating from it. Suppose the task were possible. Then the absolute values of the differences must be
1, 2, \ldots, 13, 14, so that their sum is 105, an odd number. Denoting the numbers entered in the circles by $x_i$ ($1 \leq i \leq 10$), we have that

$$105 = \sum_{i<j} |x_i - x_j| \equiv \sum_{i<j} (x_i - x_j) \equiv \sum_{i<j} (x_i + x_j) \mod 2.$$ 

In the final sum, each $x_i$ is counted as often as a segment emanates from its circle, an even number of times. This yields a contradiction. Hence the task is not possible.

5. In the figure, the side length of the large equilateral triangle is 3 and $f(3)$, the number of parallelograms bounded by sides in the grid, is 15. For the general analogous situation, find a formula for $f(n)$, the number of parallelograms, for a triangle of side length $n$.

Solution. By symmetry, the number of parallelograms is three times the number of parallelograms whose sides are parallel to the slant sides of the triangle. Suppose we enlarge the triangle by adding an additional “subbase” line with $n + 2$ dots below its base. If we extend the sides of each parallelogram to be counted, it will meet the subbase at four distinct points. Conversely, for any choice of four points from the $n + 2$ subbase points, we can form a corresponding parallelogram by drawing lines parallel to the left slant side through the left two points and lines parallel to the right slant side through the right two points. Thus, there is a one-one correspondence between parallelograms and choices of four dots. Therefore, the total number of parallelograms of all possible orientations is $3 \binom{n+2}{4}$.

The prize winners for the 1991 USA Olympiad follow. A total of 139 students from 118 schools were invited to participate.

J.P. Grossman  
Ruby Breydo  
Kiran Kedlaya  
Joel Rosenberg  
Robert Kleinberg  
Lenhard Ng  
Michail Sunitsky  
Dean Chung  

Notice that J.P. Grossman was also a winner of the CMO. Apparently the eligibility rules for the USAMO are changing next year to exclude Canadian entries.

In the April number of *Crux*, E.T.H. Wang asked if three was a record for the number of times one problem has reappeared in different Olympiads. Andy Liu responded by pointing me to the note “The art of borrowing problems” by René Laumen in the *World Federation Newsletter*, No. 6, August 1987. There Laumen discusses eight problems that
have been used more than once, one repeated three times, but three still seems to be the record.

Andy also noticed that we didn’t publish the solution to Question 2 of the 1987 Asian Pacific Mathematical Olympiad. To complete the picture he sends in the following solution. Interestingly, the published “official solution” contains an error.

Prove that the equation
\[ 6(6a^2 + 3b^2 + c^2) = 5n^2 \]
has no solution in integers except \( a = b = c = n = 0 \).

Solution by Andy Liu, University of Alberta.
We must have that \( 6 \mid n \), and then \( 3 \mid c \). Hence \( 2a^2 + b^2 + 3d^2 = 10m^2 \) where \( n = 6m \) and \( c = 3d \). If the original equation has a non-trivial solution, then this equation has one with \( \gcd(a, b, d, m) = 1 \). Clearly \( b \equiv d \mod 2 \). Now the quadratic residues modulo 16 are 0, 1, 4 and 9. The following table gives the possibilities for \( 2a^2 + b^2 + 3d^2 \mod 16 \), depending on the parities of \( a \), \( b \) and \( d \):

<table>
<thead>
<tr>
<th>( a ) odd</th>
<th>( b ) and ( d ) odd</th>
<th>( b^2 \equiv 1 ) or 9, ( 3d^2 \equiv 3 ) or 11 ( \mod 16 )</th>
<th>( b^2 \equiv 0 ) or 4, ( 3d^2 \equiv 0 ) or 12 ( \mod 16 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2a^2 \equiv 2 \mod 16 )</td>
<td>( 2a^2 + b^2 + 3d^2 \equiv 6 ) or 14 ( \mod 16 )</td>
<td>( 2a^2 + b^2 + 3d^2 \equiv 2, 6 ) or 14 ( \mod 16 )</td>
<td></td>
</tr>
<tr>
<td>( a ) even</td>
<td>( 2a^2 + b^2 + 3d^2 \equiv 4 ) or 12 ( \mod 16 )</td>
<td>( 2a^2 + b^2 + 3d^2 \equiv 0, 4, 8 ) or 12 ( \mod 16 )</td>
<td></td>
</tr>
<tr>
<td>( 2a^2 \equiv 0 ) or 8 ( \mod 16 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If \( m \) is odd then \( 10m^2 \equiv 10 \mod 16 \), and we cannot have \( 2a^2 + b^2 + 3d^2 = 10m^2 \). Hence \( m \) is even so that \( 10m^2 \equiv 0 \) or 8 \( \mod 16 \). From the above table, we must have \( a, b, d \) all even. However this contradicts that \( \gcd(a, b, d, m) = 1 \). It follows that \( 6(6a^2 + 3b^2 + c^2) = 5n^2 \) has no solution in integers except \( a = b = c = n = 0 \).

* * *

We turn now to solutions to problems from the “Archives”.

Determine the maximum value of
\[ x^3 + y^3 + z^3 - x^2y - y^2z - z^2x \]
for \( 0 \leq x, y, z \leq 1 \).

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.
Let \( f(x, y, z) \) denote the given expression. We show that \( f(x, y, z) \leq 1 \) with equality if and only if \( (x, y, z) \) equals one of the six triples: \((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1)\) and \((0, 1, 1)\).

Suppose first that \( 0 \leq x \leq y \leq z \leq 1 \). Then
\[ 1 - f(x, y, z) = (1 - z^3) + x^2(y - x) + y^2(z - y) + z^2x \geq 0 \]
with equality if and only if (i) \( z = 1 \), (ii) \( x = 0 \) or \( y = x \), (iii) \( y = 0 \) or \( z = y \), and (iv) \( z = 0 \) or \( x = 0 \). From (i) and (iv) we get \( x = 0, z = 1 \) which together with (ii) and (iii) immediately yields \((x, y, z) = (0, 0, 1)\) or \((0, 1, 1)\).

Next suppose that \( 0 \leq y \leq x \leq z \leq 1 \). Then
\[
1 - f(x, y, z) = (1 - z^3) + x(z^2 - x^2) + y(x^2 - y^2) + y^2 z \geq 0
\]
with equality just when (i) \( z = 1 \), (ii) \( x = 0 \) or \( x = z \), (iii) \( y = 0 \) or \( x = y \), and (iv) \( y = 0 \) or \( z = 0 \). From (i) and (iv) we conclude that \( z = 1, y = 0 \). Now (ii) gives the two solutions \((x, y, z) = (0, 0, 1)\) and \((1, 0, 1)\).

Exploiting the cyclic symmetry of \( f(x, y, z) \) we may reduce to one of these two cases, and this gives the six solutions listed.

\[
\begin{align*}
1 - f(x, y, z) &= (1 - z^3) + x(z^2 - x^2) + y(x^2 - y^2) + y^2 z \\
&\geq 0
\end{align*}
\]

\[
(x, y, z) = (0, 0, 1) \text{ or } (1, 0, 1).
\]

For \( s, t \in \mathbb{N} \), let
\[
M = \{ (x, y) \mid 1 \leq x \leq s, \ 1 \leq y \leq t, \ x, y \in \mathbb{N} \}
\]
be a given set of points in a plane. Determine the number of rhombuses whose vertices belong to \( M \) and whose diagonals are parallel to the \( x \), \( y \) coordinate axes.

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

The diagonals of the rhombuses must have even lengths. Consider \( t \) points spaced at unit intervals on a line. In how many ways can one select two points at even distance? There are \( t - 2 \) pairs with distance 2, \( t - 4 \) pairs with distance 4, and so on. This sums to give \([t - 1]^2/4\] where \([x]\) is the integer part of \( x \). One gets all rhombuses by combining every \( t \)-pair with every \( s \)-pair giving
\[
\left\lfloor \frac{(t - 1)^2}{4} \right\rfloor \cdot \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor.
\]

Determine the set of all values of \( x_0, x_1 \in \mathbb{R} \) such that the sequence defined by
\[
x_{n+1} = \frac{x_n - 1 \cdot x_n}{3x_{n-1} - 2x_n}, \quad n \geq 1
\]
contains infinitely many natural numbers.


From the recurrence relation, \( x_n = 0 \) gives \( x_{n+1} = 0 \) and the sequence breaks down. So we assume \( x_n \neq 0 \). Now put \( u_n = 1/x_n \). The recurrence relation now gives \( u_{n+2} - 3u_{n+1} + 2u_n = 0 \), with characteristic equation \( t^2 - 3t + 2 = 0 \), having 1 and 2 as its roots. Hence \( u_n = a + 2^n b \), where \( a \) and \( b \) do not depend on \( n \). If \( b \neq 0 \) we get \( \lim_{n \to \infty} |u_n| = \infty \) and so \( x_n \) tends to zero in which case the sequence is an integer only finitely often. Thus \( u_n \) and hence \( x_n \) are constants. The answer is then \( x_0 = x_1 \in \mathbb{N} - \{0\} \).
Determine all quadruples \((x, y, u, v)\) of real numbers satisfying the simultaneous equations
\[
\begin{align*}
x^2 + y^2 + u^2 + v^2 &= 4, \\
xu + yv &= -vx - yu, \\
xy + yu + uv + vx &= -2, \\
xyuv &= -1.
\end{align*}
\]

*Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.*

The second equation can be rewritten
\[
(x + y)(u + v) = 0
\]
and the third is
\[
xy(u + v) + (x + y)uv = -2.
\]

*Case 1.* \(x + y = 0\). Then the last two equations give \(x^2(u + v) = 2\) and \(x^2uv = 1\).

From this \(u + v = 2uv\) and \(x^2 = y^2 = 2/(u + v)\). Using the first equation,
\[
\frac{4}{u + v} + (u + v)^2 - (u + v) = 4.
\]

Setting \(z = u + v\) we obtain
\[
(z - 1)(z - 2)(z + 2) = z^3 - z^2 - 4z + 4 = 0.
\]

This now gives three subcases.

*Subcase (i):* \(u + v = 1 = 2uv\), no real solution.

*Subcase (ii):* \(u + v = 2 = 2uv\), giving \(u = 1, v = 1, x = \pm 1, y = \pm 1\) and solutions
\((1, -1, 1, 1), (-1, 1, 1, 1)\).

*Subcase (iii):* \(u + v = -2 = 2uv\), and no real solution for \(x\).

*Case 2.* \(u + v = 0\). By symmetry we obtain \((1, 1, -1, 1)\) and \((1, 1, 1, -1)\).


An \(m \times n\) matrix of distinct real numbers is given. The elements of each row are rearranged (in the same row) such that the elements are increasing from left to right. Next the elements of each column are rearranged (in the same column) such that the elements are increasing from top to bottom. Show that now the elements in each row are still increasing from left to right.

*Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

This problem is essentially the same as question 2 on the 1980 Canadian Mathematics Olympiad where a \(5 \times 10\) matrix was considered instead of an \(m \times n\) matrix. The published official solution which was actually for the general case can be found on page 26 in “Report of the Twelfth CMO” (published by the CMS) [and also in *Crux* [1980: 241]].
Now we turn to the November 1989 number of *Crux* and the rest of the problems proposed but not used at the 1989 IMO. The “official” solutions can be found in the booklet 30th International Mathematical Olympiad, Braunschweig, 1989, edited by Hanns-Heinrich Langmann.


Ali Baba the carpet merchant has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. He knows that the sides of the rooms are integral numbers of feet, and that his two storerooms have the same (unknown) length, but widths of 38 and 55 feet respectively. What are the carpet’s dimensions?

*Correction by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.*

Let the carpet have length $y$, width $x$. Let the length of the storerooms be $L$, and set $y/x = k$. The diagram shows the larger storoom. By similar triangles, $BE/y = b/x$ or $BE = kb$. Similarly $DF = ka$. Now $a + kb = 55$ and $ka + b = L$, so

$$a = \frac{kL - 55}{k^2 - 1}, \quad b = \frac{55k - L}{k^2 - 1}.$$

This gives

$$x^2 = \left(\frac{kL - 55}{k^2 - 1}\right)^2 + \left(\frac{55k - L}{k^2 - 1}\right)^2$$

so

$$(k^2 - 1)^2 x^2 = (kL - 55)^2 + (55k - L)^2.$$

In the same way, the other room yields

$$(k^2 - 1)^2 x^2 = (kL - 38)^2 + (38k - L)^2.$$

Equating we have

$$(38k - L)^2 + (Lk - 38)^2 = (55k - L)^2 + (Lk - 55)^2.$$

The discriminant of this equation (after removing square factors) is $D = 4L^2 - 93^2 \equiv 3 \mod 4$, so this equation has no rational solution.

There must be a misprint in the problem and it is likely that the widths should be 38 and 50 instead of 38 and 55. The equation then becomes

$$(38k - L)^2 + (Lk - 38)^2 = (50k - L)^2 + (Lk - 50)^2$$

which simplifies to $22k^2 - Lk + 22 = 0$. The discriminant is $D = L^2 - 44^2$ and this gives rise to rational solutions.
Editor’s Note. The “official” solution in the IMO booklet starts out with the assumption that the widths are 38 and 50, even though the problem is stated as above, with widths of 38 and 55!


At $n$ distinct points of a circle-shaped race course there are $n$ cars ready to start. They cover the circle in an hour. Hearing the signal each of them selects a direction and starts immediately. If two cars meet both of them change directions and go on without loss of speed. Show that at a certain moment each car will be at its starting point.

Solution by Curtis Cooper, Central Missouri State University, Warrensburg.

We will show that after $n$ hours, each car will be back at its starting point. Consider another car race such that if two cars meet they pass through each other instead of changing directions. Although the races are different, the “passing through” car race and the “changing direction” car race are similar in that at any given time, the points on the course occupied by cars are the same and the directions the cars at these points are moving are the same. After one hour of the “passing through” car race, each car is back at its starting point and going in its starting direction. Thus, although the cars may not be the same, we see that after one hour of the “changing direction” car race, there will be a car at each starting point and that this car will be going in the same direction that the starting car at that point went. In addition, although we may not know which car is at each starting point in one hour, we do know that the cars in the “changing direction” car race maintain the same relationship with the other cars. That is, in the “changing direction” car race, the cars in the clockwise direction and counterclockwise direction with respect to a given car are always the same. Therefore, after one hour of the “changing direction” race, the cars at the starting points on the course are some rotation (possibly $0^\circ$) of the original cars at the starting points on the course and are going in the same directions the original cars were going. Thus after $n$ hours, the cars in the “changing direction” car race will be back at the starting point.

* * *

In the remaining space for this month’s number of the Corner, we given solutions sent in by readers to problems of the 11th Austrian–Polish Mathematics Competition that was given in the December 1989 number of Crux [1989: 289–291].

1. Let $P(x)$ be a polynomial having integer coefficients. Show that if $Q(x) = P(x) + 12$ has at least six distinct integer roots, then $P(x)$ has no integer roots.


Define $M(2k) = (k!)^2$ and $M(2k + 1) = k!(k + 1)!$, where $k$ is a positive integer. We show:

THEOREM. If $Q(x)$ has integer coefficients and $k$ distinct integer roots and if $P(x) = Q(x) - y$ has an integral root with $y \neq 0$, then $|y| \geq M(k)$.

The given problem has $k = 6$, $y = 12$, so that $M(k) = 36 \leq |y|$, and the problem follows.
To prove the theorem, let the given roots of $Q(x)$ be $x_1, x_2, \ldots, x_k$ and let

$$B(x) = (x - x_1)(x - x_2)\ldots(x - x_k).$$

We know $Q(x)$ is a multiple of $B(x)$, say $Q(x) = A(x)B(x)$.

**LEMMA.** $A(x)$ has integral coefficients.

**Proof.** Let

$$A(x) = \sum_{i=0}^{n} a_ix^i, \quad B(x) = \sum_{i=0}^{k} b_ix^i, \quad Q(x) = \sum_{i=0}^{n+k} q_ix^i.$$

We know $q_i \in \mathbb{Z}$, $b_i \in \mathbb{Z}$ and $b_k = 1$. Thus $q_{n+k} = a_nb_k$ implies $a_n \in \mathbb{Z}$. Then $q_{n+k-1} = a_nb_{k-1} + a_{n-1}b_k$ gives us $a_{n-1} = q_{n+k-1} - a_nb_{k-1}$ and $a_{n-1} \in \mathbb{Z}$. Etc. □

Now suppose $P(x) = Q(x) - y$ has an integral root $x_0$. Since $y \neq 0$, $x_0$ is distinct from all the $x_i$, and $P(x_0) = Q(x_0) - y = A(x_0)B(x_0) - y = 0$, so $y = A(x_0)B(x_0)$. From the lemma, we know $A(x_0)$ is an integer, so

$$B(x_0) = (x_0 - x_1)\ldots(x_0 - x_k)|y.$$  \hspace{1cm} (1)

The values $x_0 - x_i$ are $k$ distinct integers so the least absolute value of such a product is $1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdots$ (with $k$ terms) = $M(k)$; hence $|y| \geq M(k)$. This proves the theorem.

In place of 12, the original problem could have had any integer in the range $[1, 35]$.

*Editor’s Note.* The problem was also solved by Seung-Jin Bang, Seoul, Republic of Korea; by Curtis Cooper, Central Missouri State University, Warrensburg; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

4. Determine all strictly monotone increasing functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the functional equation

$$f(f(x) + y) = f(x + y) + f(0)$$

for all $x, y \in \mathbb{R}$.

*Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by George Evangelopoulos, Athens, Greece; and by David C. Vaughan, Wilfrid Laurier University, Waterloo, Ontario.

Let $f$ be a strictly increasing function on $\mathbb{R}$ satisfying the functional equation. In particular, if we set $y = -x$, we must have $f(f(x) - x) = f(0) + f(0) = 2f(0)$. Unless $f(x) - x$ is a constant we have a contradiction to the fact that $f$ is $1 - 1$ since $f(f(x) - x)$ is constant. Thus $f(x) = x + c$ where $c = f(0)$. Any such $f$ is strictly increasing, and we have

$$f(f(x) + y) = f(x + c + y) = (x + y + c) + c = f(x + y) + f(0),$$

that is, any such $f$ is a solution.

7. In a regular octagon each side is coloured blue or yellow. From such a colouring another colouring will be obtained “in one step” as follows: if the two neighbours of a side have different colours, the “new” colour of the side will be blue, otherwise the colour will be yellow. [Editor’s note: the colours are modified simultaneously.] Show that after a
finite number, say $N$, of moves all sides will be coloured yellow. What is the least value of $N$ that works for all possible colourings?

Solution by Curtis Cooper, Central Missouri State University.

Consider a colouring of the sides of a regular octagon where each side is coloured blue or yellow. Label the sides of the octagon $S_1, \ldots, S_8$ and code the colours of the sides of the regular octagon by the column vector $x = (x_1, \ldots, x_8)$, where

$$x_i = \begin{cases} 
1 & \text{if } S_i \text{ is coloured blue,} \\
0 & \text{if } S_i \text{ is coloured yellow,}
\end{cases}$$

for $i = 1, 2, \ldots, 8$. Let

$$A = \begin{pmatrix} 
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$ 

Then $A^n x$ is the coded colouring of the sides of the regular octagon after $n$ steps, where the addition and multiplication in the matrix operations are performed mod 2. Since $A^4 = 0$ it follows that after 4 steps all sides will be coloured yellow. This is the least possible value of $N$ that works for all possible colourings, since for $x = (1, 0, 0, 0, 0, 0, 0, 0)^T$,

$$A x = (0, 1, 0, 0, 0, 0, 0, 1)^T, \quad A^2 x = (0, 0, 1, 0, 0, 1, 0, 1)^T, \quad A^3 x = (0, 1, 0, 1, 0, 1, 0, 1)^T.$$

* * * *

This completes the space for this month. Send in your nice solutions and your Olympiads.

* * * * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.
To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1992, although solutions received after that date will also be considered until the time when a solution is published.

1661. Proposed by Toshio Seimiya, Kawasaki, Japan.
\[ \triangle ABC \text{ is inscribed in a circle } \odot. \text{ Let } D \text{ be a point on } BC \text{ produced such that } AD \text{ is tangent to } \odot \text{ at } A. \text{ Let } \odot 2 \text{ be a circle which passes through } A \text{ and } D, \text{ and is tangent to } BD \text{ at } D. \text{ Let } E \text{ be the point of intersection of } \odot 1 \text{ and } \odot 2 \text{ other than } A. \text{ Prove that } E B : E C = AB^3 : AC^3. \]

1662. Proposed by Murray S. Klamkin, University of Alberta.
Prove that
\[ \frac{x^{2r+1}_1}{s-x_1} + \frac{x^{2r+1}_2}{s-x_2} + \cdots + \frac{x^{2r+1}_n}{s-x_n} \geq \frac{4^r}{(n-1)n^{2r-1}}(x_1x_2 + x_2x_3 + \cdots + x_nx_1)^r, \]
where \( n > 3, r \geq 1/2, x_i \geq 0 \) for all \( i \), and \( s = x_1 + x_2 + \cdots + x_n \). Also, find some values of \( n \) and \( r \) such that the inequality is sharp.

1663\textsuperscript{*}. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.
Let \( A, B, C \) be the angles of a triangle, \( r \) its inradius and \( s \) its semiperimeter. Prove that
\[ \sum \sqrt{\cot(A/2)} \leq \frac{3}{2} \sqrt{r/s} \sum \csc(A/2), \]
where the sums are cyclic over \( A, B, C \).

1664. Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.
(Dedicated to Jack Garfunkel.)
Consider two concentric circles with radii \( R_1 \) and \( R \) \( (R_1 > R) \) and a triangle \( ABC \) inscribed in the inner circle. Points \( A_1, B_1, C_1 \) on the outer circle are determined by extending \( BC, CA, AB \), respectively. Prove that
\[ \frac{F_1}{R_1^2} \geq \frac{F}{R^2}, \]
where \( F_1 \) and \( F \) are the areas of triangles \( A_1B_1C_1 \) and \( ABC \) respectively, with equality when \( ABC \) is equilateral.

1665\textsuperscript{*} ends in 5 for \( n \geq 1 \), and in 25 for \( n \geq 2 \). Find the longest string of digits which ends 1665\textsuperscript{*} for all sufficiently large \( n \).

1666. Proposed by Marcin E. Kuczma, Warszawa, Poland.
(a) How many ways are there to select and draw a triangle \( T \) and a quadrilateral \( Q \) around a common incircle of unit radius so that the area of \( T \cap Q \) is as small as possible? (Rotations and reflections of the figure are not considered different.)
(b) The same question, with the triangle and quadrilateral replaced by an \( m \)-gon and an \( n \)-gon, where \( m, n \geq 3 \).
Evaluate
\[ \sum_{n=1}^{\infty} \coth^{-1}(2^{n+1} - 2^{-n}). \]

1668. Proposed by Stanley Rabinowitz, Westford, Massachusetts.
What is the envelope of the ellipses
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
as \(a\) and \(b\) vary so that \(a^2 + b^2 = 1\)?

1669. Proposed by J.P. Jones, University of Calgary.
Suppose that \(q\) is a rational number with \(|q| \leq 2\), and that
\[ q + i\sqrt{4 - q^2} \]
is an \(n\)th root of unity for some \(n\). Show that \(q\) must be an integer.

1670* Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.

Let \(B_1, B_2, C_1, C_2\) be points in the plane and let lines \(B_1B_2\) and \(C_1C_2\) intersect in \(A\). Prove that the four points \(G_{11}, G_{12}, G_{21}, G_{22}\) form the vertices of a parallelogram when \(G_{ij}\) is determined in any of the following ways: (i) \(G_{ij}\) is the centroid of \(\Delta AB_iC_j\); (ii) \(G_{ij}\) is the orthocenter of \(\Delta AB_iC_j\); (iii) \(G_{ij}\) is the circumcenter of \(\Delta AB_iC_j\).

* * * * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

For fixed \(n\), determine the minimum value of
\[ C_n = |\cos \theta| + |\cos 2\theta| + \cdots + |\cos n\theta|. \]
It is conjectured that \(\min C_n = \lceil n/2 \rceil\) for \(n > 2\).

Solution by Manuel Benito Muñoz and Emilio Fernández Moral, I.B. Sagasta, Logroño, and María Mercedes Sánchez Benito, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.
We will prove that the conjecture is true for all \(n > 2\) except for \(n = 4\) and \(n = 6\). Consider the function
\[ F_n(x) = \sum_{j=1}^{n} \left| \cos \left( jx \frac{\pi}{2} \right) \right|. \]
We will first prove that

\((*)\) for \(n \geq 8\) with \(n\) even, \(F_n(x) \geq n/2\);

this proves the conjecture for all \(n \geq 8\), since then

\[
\min F_{n+1}(x) \geq \min F_n(x) = \frac{n}{2} = \left\lfloor \frac{n+1}{2} \right\rfloor,
\]

and we have

\[
F_{n+1}(1) = F_n(1) + \left| \cos \left( \frac{(n+1)\pi}{2} \right) \right| = F_n(1) = \frac{n}{2}.
\]

Finally, we verify that the conjecture is true for \(n = 3, 5\) and \(7\):

\[F_3(x) \geq F_3(1) = 1, \quad F_5(x) \geq F_5(1) = 2, \quad F_7(x) \geq F_7(1) = 3.\]

However the minimum values for \(n = 4\) and \(n = 6\) are respectively

\[
\min F_4(x) = F_4(1/3) = 1 + \frac{\sqrt{3}}{2} < 2,
\]

\[
\min F_6(x) = F_6(1/5) \approx 2.97 < 3.
\]

We begin with the following lemma.

**LEMMA 1.** (a) \(F_n(x + 2) = F_n(x)\). (This allows us to study the function on \([0, 2]\) only.)

(b) \(F_n(1 - x) = F_n(1 + x)\). (This allows us to study the function on \([0, 1]\) only.)

(c) \(F_n(x)\) attains its relative minimum values (in the interval \([0, 1]\)) on points of the Farey sequence of order \(n\) (i.e., irreducible fractions in \([0, 1]\) with denominator at most \(n\)) having odd numerator.

**Proof.** (a) Follows from

\[
\left| \cos \frac{j(x + 2)\pi}{2} \right| = \left| \cos \left( \frac{jx\pi}{2} + j\pi \right) \right| = \left| \cos \frac{jx\pi}{2} \right|.
\]

(b) Follows from

\[
\left| \cos \frac{j(1 - x)\pi}{2} \right| = \left| \cos \left( \frac{j\pi}{2} - \frac{jx\pi}{2} \right) \right| = \left| \cos \left( \frac{j\pi}{2} + \frac{jx\pi}{2} \right) \right|.
\]

(c) \(F_n(x)\) has a derivative at all \(x\) except those \(x\) for which \(\cos(jx\pi/2) = 0\) for some \(j \in \{1, 2, \ldots, n\}\). On the interval \([0, 1]\), these points will be of the form \(x = (2k + 1)/j\) with \(j = 1, 2, \ldots, n\) and \(k\) an integer with \(0 < 2k + 1 < j\), that is, fractions of the Farey sequence of order \(n\) with odd numerator. If \(a/b, c/d\) are two adjacent fractions (with odd numerator) in the Farey sequence of order \(n\), \(F_n(x)\) is differentiable on the interval \((a/b, c/d)\) and, if we call \(\text{sgn}(j)\) the constant sign of \(\cos(jx\pi/2)\) on this interval, then

\[
F_n'(x) = -\sum_{j=1}^{n} \text{sgn}(j) \cdot \frac{j\pi}{2} \sin \left( \frac{jx\pi}{2} \right),
\]
\[ F''_n(x) = -\sum_{j=1}^{\infty} \text{sgn}(j) \cdot \frac{j^2 \pi^2}{4} \cos \left( \frac{3j\pi x}{2} \right) = -\frac{\pi^2}{4} \sum_{j=1}^{\infty} j^2 \left| \cos \frac{j\pi x}{2} \right| < 0. \]

Hence \( F_n(x) \) can have only relative maximum points inside the interval, and the minimum value on \([a/b, c/d]\) will be attained at one of the endpoints. \( \square \)

By this lemma, to prove \( F_n(x) \geq n/2 \) it suffices to verify that \( F_n(a/b) \geq n/2 \) for \( 0 < a < b \leq n, \) a odd and a prime to \( b. \) (It is clear that \( F_n(0) \) and \( F_n(1) \) are both \( \geq n/2 \).)

**LEMMA 2.** If \( a \) and \( b \) are positive integers with \( a \) odd and prime to \( b, \) then \( F_b(a/b) = F_b(1/b). \)

**Proof.** If \( a \) is odd and prime to \( b, \) then \( a \) is prime to \( 2b. \) Thus for \( j = 1, 2, \ldots, b, \) we can write \( ja = 2bq_j + k_j \) with \( q_j, k_j \) integers, \( 1 \leq |k_j| \leq b. \) It is easy to see that \( \{ |k_j| : j = 1, 2, \ldots, b \} = \{ 1, 2, \ldots, b \} \) if \( |k_j| = |k_1|, \) then \( ja - 2bq_j = \pm (\ell a - 2bq_\ell), \) so
\[ 2b(q_j + q_\ell) = a(j \pm \ell); \] now \( (a, 2b) = 1 \) means \( 2b(j \pm \ell) \) which is impossible if \( j \neq \ell. \) Consequently
\[
F_b \left( \frac{a}{b} \right) = \sum_{j=1}^{b} \left| \cos \frac{j\pi a}{2b} \right| = \sum_{j=1}^{b} \left| \cos \frac{2bq_j + k_j}{2b} \right| = \sum_{j=1}^{b} \left| \cos \left( q_j \frac{\pi}{2} + \frac{k_j}{2b} \right) \right|
\]
\[ = \sum_{j=1}^{b} \left| \cos \frac{k_j}{2b} \right| = \sum_{j=1}^{b} \left| \cos \frac{j\pi}{2b} \right| = \sum_{j=1}^{b} \left| \cos \frac{j}{2} \right| = F_b \left( \frac{1}{b} \right). \quad \square \]

As we shall see later, this lemma will reduce our proofs to searching the values of \( F_n(x) \) for \( x = 1/b \) with \( b \leq n. \)

Till now we had in mind that \( n \) was a fixed number and \( b \) was a variable number less than \( n. \) Now we must change this point of view, and we consider \( b \geq 3 \) fixed and \( n \) variable.

**PROPOSITION 1.** Let \( b \geq 3 \) be an integer. If \( b \leq n < 2b \) with \( n \) even, then \( F_n(1/b) > n/2 \) except for the cases
\[ F_3(1/3) \approx 1.87 < 2 \quad \text{and} \quad F_6(1/5) \approx 2.97 < 3. \]

**Proof.** \( F_n(1/b) > n/2 \) is equivalent to: the average of the terms
\[ \left| \cos \frac{j\pi}{2b} \right|, \quad j = 1, 2, \ldots, n, \]

is greater than \( 1/2. \) Note that
\[
\left| \cos \frac{j\pi}{2b} \right| \begin{cases} 
> 1/2 & \text{for } 1 \leq j < 2b/3, \\
< 1/2 & \text{for } 2b/3 < j < 4b/3, \\
> 1/2 & \text{for } 4b/3 < j < n.
\end{cases}
\]

Thus it suffices to show that \( F_n(1/b) > n/2 \) for \( n = \) whichever of \( [4b/3] - 1, \) \([4b/3], \) \([4b/3] + 1 \) are even, that is,

(i) for \( b = 3c, \) \( F_{3c}(1/b) > 2c; \)
(ii) for \( b = 3c + 1 \), \( F_{4c}(1/b) > 2c \) and \( F_{4c+2}(1/b) > 2c + 1 \);

(iii) for \( b = 3c + 2 \), \( F_{4c+2}(1/b) > 2c + 1 \).

Since the function \(|\cos(\pi x/2)|\) is concave, we can bound it on \([0, 2]\) from below by the polygonal path obtained by connecting the points \((x, |\cos(\pi x/2)|)\) for \( x = 0, 1/3, 2/3, 1, 4/3, 5/3, 2\) by straight lines. This gives a function \(H(x)\) defined by

\[
H(x) = \begin{cases} 
1 + \left(\frac{3\pi - 6}{2}\right) x & = u_0 + v_0 x \quad \text{for } x \in [0, 1/3], \\
\left(\sqrt{3} - \frac{1}{2}\right) + \left(\frac{3 \sqrt{3} - 3}{2}\right) x & = u_1 + v_1 x \quad \text{for } x \in [1/3, 2/3], \\
\frac{3}{2} + (-3/2)x & = u_2 + v_2 x \quad \text{for } x \in [2/3, 1], \\
-3/2 + (3/2)x & = u_3 + v_3 x \quad \text{for } x \in [1, 4/3], \\
\left(\frac{5-4\sqrt{3}}{2}\right) + \left(\frac{3\pi - 3}{2}\right) x & = u_4 + v_4 x \quad \text{for } x \in [4/3, 5/3], \\
(3\sqrt{3} - 5) + \left(\frac{6 - 3\sqrt{3}}{2}\right) x & = u_5 + v_5 x \quad \text{for } x \in [3/2, 2].
\end{cases}
\]

For (i),

\[
F_{4c} \left(\frac{1}{3c}\right) = \sum_{j=1}^{4c} \left| \cos \left(\frac{j\pi}{6c}\right) \right| \geq \sum_{j=1}^{4c} H \left(\frac{j}{3c}\right) = \sum_{i=0}^{3} \sum_{j=i+1}^{(i+1)c} \left( u_i + v_i \frac{j}{3c} \right) \\
= c \sum_{i=0}^{3} u_i + \frac{1}{3c} \sum_{i=0}^{3} v_i \frac{c(2i + 1)c + 1}{2} \\
= c \left(\sqrt{3} + \frac{1}{2}\right) + \frac{c + 1}{6} \sum_{i=0}^{3} v_i + \frac{c}{3} \sum_{i=0}^{3} iv_i \\
= c \left(\sqrt{3} + \frac{1}{2}\right) + \frac{c + 1}{6} \left(\frac{3}{2}\right) + \frac{c}{3} \left(\frac{6 - 3\sqrt{3}}{2}\right) \\
= \frac{c(5 + 2\sqrt{3}) - 1}{4} = 2c + \frac{(2\sqrt{3} - 3)c - 1}{4} > 2c \quad \text{for } c \geq 3,
\]

while for \( c = 2 \), by direct calculation,

\[
F_8(1/6) = \sum_{j=1}^{8} \left| \cos \left(\frac{j\pi}{12}\right) \right| \approx 4.05 > 4.
\]

For \( c = 1 \) appears the first special case: direct computation shows that

\[
F_4(1/3) = 1 + \frac{\sqrt{3}}{2} < 2.
\]

For (ii),

\[
F_{4c} \left(\frac{1}{3c+1}\right) \geq \sum_{j=1}^{c} \left( u_0 + v_0 \frac{j}{3c+1} \right) + \sum_{j=c+1}^{2c} \left( u_1 + v_1 \frac{j}{3c+1} \right) + \sum_{j=2c+1}^{3c+1} \left( u_2 + v_2 \frac{j}{3c+1} \right) + \sum_{j=3c+2}^{4c} \left( u_3 + v_3 \frac{j}{3c+1} \right)
\]
\[
= c \sum_{i=0}^{3} u_i + u_2 - u_3 + \frac{1}{6c+2} [c(c+1)v_0 + (3c+1)c v_1 + (5c+2)(c+1)v_2 \\
+ (7c+2)(c-1)v_3] \\
= \frac{(15 + 6\sqrt{3})c^2 + (4\sqrt{3} - 1)c}{12c+4} = 2c + \frac{(6\sqrt{3} - 9)c^2 - (9 - 4\sqrt{3})c}{12c+4} \\
> 2c \quad \text{for } c \geq 2,
\]
while for \( c = 1 \), \( F_4(1/4) > 2 \). Also, from the above calculation,
\[
F_{4c+2} \left( \frac{1}{3c+1} \right) \geq 2c + \frac{(6\sqrt{3} - 9)c^2 - (9 - 4\sqrt{3})c}{12c+4} + \frac{3}{2b} (4c+1) \quad \text{for } c \geq 1.
\]
For (iii),
\[
F_{4c+2} \left( \frac{1}{3c+2} \right) \geq \sum_{j=1}^{c} \left( u_0 + u_0 \frac{j}{3c+2} \right) + \sum_{j=1}^{2c+1} \left( u_1 + v_1 \frac{j}{3c+2} \right) \\
+ \sum_{j=2c+2}^{3c+2} \left( u_2 + v_2 \frac{j}{3c+2} \right) + \sum_{j=3c+3}^{4c+2} \left( u_3 + v_3 \frac{j}{3c+2} \right) \\
= c \sum_{i=0}^{3} u_i + u_1 + u_2 + \frac{1}{6c+4} [c(c+1)v_0 + (3c+2)(c+1)v_1 \\
+ (5c+4)(c+1)v_2 + (7c+5)cv_3] \\
= 2c + 1 + \frac{(6\sqrt{3} - 9)c^2 + (8\sqrt{3} - 15)c - (6 - 2\sqrt{3})}{12c+8} \\
> 2c + 1 \quad \text{for } c \geq 2.
\]
When \( c = 1 \) we have the other exceptional case, because
\[
F_6(1/5) \approx 2.97 < 3.
\]
Thus, Proposition 1 is proved. \( \square \)

PROPOSITION 2. Let \( a \) and \( b \) be positive integers such that \( a < b \), \( a \) is odd, and \( \gcd(a, b) = 1 \). Then for all \( n \) even such that \( b \leq n < 2b \), \( F_n(a/b) \geq F_n(1/b) \).

Proof. According to Lemma 2,
\[
F_n \left( \frac{a}{b} \right) = F_b \left( \frac{a}{b} \right) + \sum_{j=b+1}^{n} \left| \cos \left( \frac{ja\pi}{2b} \right) \right| = F_b \left( \frac{1}{b} \right) + \sum_{j=b+1}^{n} \left| \cos \left( \frac{ja\pi}{2b} \right) \right|.
\]
For each \( j = b + 1, \ldots, n \), find \( k_j \) such that \( b + 1 \leq |k_j| \leq 2b - 1 \) and \( ja \equiv k_j \mod 2b \). As in the proof of Lemma 2, the \( k_j \)'s exist, and the numbers \( |k_j|, j = b + 1, \ldots, n \), are distinct. Also as in Lemma 2, for all \( j \),

\[
|\cos \left( \frac{ja\pi}{2b} \right)| = \left| \cos \left( \frac{|k_j|\pi}{2b} \right) \right|.
\]

Thus in the summation

\[
\sum_{j = b + 1}^{n} \left| \cos \left( \frac{ja\pi}{2b} \right) \right|
\]

we are adding \( n - b \) elements of the set

\[
\left\{ \left| \cos \left( \frac{j\pi}{2b} \right) \right| : b + 1 \leq j \leq 2b - 1 \right\};
\]

but the smallest such sum is

\[
\sum_{j = b + 1}^{n} \left| \cos \left( \frac{j\pi}{2b} \right) \right|;
\]

formed by the smallest \( n - b \) terms. Thus

\[
F_n \left( \frac{a}{b} \right) \geq F_b \left( \frac{1}{b} \right) + \sum_{j = b + 1}^{n} \left| \cos \left( \frac{j\pi}{2b} \right) \right| = F_n \left( \frac{1}{b} \right). \quad \Box
\]

Our next step is to study the case \( 2b \leq n < 3b \).

**PROPOSITION 3.** Let \( a \) and \( b \) be positive integers where \( a < b \), \( a \) is odd, and \( \gcd(a, b) = 1 \). Then for all even \( n \) such that \( 2b \leq n < 3b \), \( F_n(a/b) > n/2 \).

**Proof.** We use the notation from the proof of Proposition 1. Note first that, by the symmetry of \( F_n(x) \) and by Lemma 2,

\[
F_{2b} \left( \frac{a}{b} \right) = \sum_{j = 1}^{b-1} \left| \cos \frac{ja\pi}{2b} \right| + 0 + \sum_{j = b+1}^{2b-1} \left| \cos \frac{ja\pi}{2b} \right| + 1
\]

\[
= 2F_b \left( \frac{1}{b} \right) + 1 = 2F_b \left( \frac{1}{b} \right) + 1 = F_{2b} \left( \frac{1}{b} \right).
\]

**Case (i):** \( b = 3c \). Then

\[
F_{6c} \left( \frac{1}{3c} \right) \geq \sum_{i=0}^{5} \sum_{j=0}^{c} \left( u_i + v_i j \right) = c \sum_{i=0}^{5} u_i + \frac{c + 1}{6} \sum_{i=0}^{5} v_i + \frac{c}{3} \sum_{i=0}^{5} iv_i
\]

\[
= c(2\sqrt{3} - 2) + 0 + \frac{c}{3}(12 - 3\sqrt{3}) = 3c + (\sqrt{3} - 1)c,
\]

and since \( 6c < n < 9c \),

\[
F_n \left( \frac{a}{3c} \right) = F_{6c} \left( \frac{a}{3c} \right) + \sum_{j=6c+1}^{n} \left| \cos \frac{ja\pi}{6c} \right|.
\]
But all terms \(|\cos(ja\pi/6c)|\) of the above sum are greater than or equal to 1/2, except for at most \(c\) of them. [Editor’s note. In general, \(|\cos(ja\pi/2b)| < 1/2\) for \(2b \leq j \leq n\) if and only if \(j\) belongs to an interval of the form

\[
\left(\frac{2b(t + a)}{3a}, \frac{2b(t + a)}{3a} + \frac{4b}{3a}\right)
\]

for some nonnegative integer \(t\), where \(n < 3b\) means \(t < a/2\); thus there are less than \(a/2\) such intervals, each containing at most \(2b/3a\) such \(j\)’s, so at most \(b/3\) such \(j\)’s altogether.] Hence from (1), (2) and (3),

\[
F_n\left(\frac{a}{3c}\right) \geq 3c + (\sqrt{3} - 1)c + (n - 7c) \cdot \frac{1}{2} = n + \left(\sqrt{3} - \frac{3}{2}\right)c > \frac{n}{2}.
\]

Case (ii): \(b = 3c + 1\). As in the case above, we have

\[
F_{6c+2}\left(\frac{a}{3c + 1}\right) = F_{6c+2}\left(\frac{1}{3c + 1}\right)
\]

\[
\geq \sum_{j=1}^{c} \left( u_0 + v_0 \frac{j}{3c + 1} \right) + \sum_{j=c+1}^{2c} \left( u_1 + v_1 \frac{j}{3c + 1} \right)
\]

\[
+ \sum_{j=2c+1}^{3c+1} \left( u_2 + v_2 \frac{j}{3c + 1} \right) + \sum_{j=3c+2}^{4c+1} \left( u_3 + v_3 \frac{j}{3c + 1} \right)
\]

\[
+ \sum_{j=4c+2}^{5c+1} \left( u_4 + v_4 \frac{j}{3c + 1} \right) + \sum_{j=5c+2}^{6c+2} \left( u_5 + v_5 \frac{j}{3c + 1} \right)
\]

\[
= c \sum_{i=0}^{5} u_i + u_2 + u_5 + \frac{1}{6c + 2} [c(c + 1)v_0 + c(3c + 1)v_1 + (c + 1)(5c + 2)v_2
\]

\[
+ c(7c + 3)v_3 + c(9c + 3)v_4 + (c + 1)(11c + 4)v_5]
\]

\[
= \frac{(6 + 3\sqrt{3})c^2 + (4 + 2\sqrt{3})c + 1}{3c + 1} = 3c + 1 + (\sqrt{3} - 1)c \left(1 + \frac{1}{3c + 1}\right)
\]

\[> 3c + 1 + (\sqrt{3} - 1)c.\]

Now since \(6c + 2 < n < 9c + 3\),

\[
F_n\left(\frac{a}{3c + 1}\right) = F_{6c+2}\left(\frac{a}{3c + 1}\right) + \sum_{j=6c+3}^{n} \left|\cos \frac{j\pi}{6c + 2}\right|
\]

\[
> 3c + 1 + (\sqrt{3} - 1)c + \frac{n - 7c - 2}{2} > \frac{n}{2},
\]

because, as before [see the editor’s note in case (i)], of the \(n - 6c - 2\) terms in the summation, at most \(c\) will be less than \(1/2\).

Case (iii): \(b = 3c + 2\). In the same way,

\[
F_{6c+4}\left(\frac{a}{3c + 2}\right) = F_{6c+4}\left(\frac{1}{3c + 2}\right)
\]
If we suppose the proposition is true when \((2k-1)b \leq n < (2k+1)b\), then when \((2k+1)b \leq n < (2k+3)b\) we obtain by (4) and Proposition 3 that

\[ F_n \left( \frac{a}{b} \right) > b + \frac{n - 2b}{2} = \frac{n}{2}, \]

so the proposition follows provided we can establish the base case. If \(b \neq 2, 3\) or \(5\) the proposition is true for \(k = 1\), i.e., in the case \(b \leq n < 3b\), by Propositions 1, 2 and 3.
Finally, for the remaining cases $b = 2, 3$ and $5$ we could begin the induction on the interval $3b \leq n < 5b$, since by direct computation

$$F_6(1/2) \approx 3.1 > 3, \quad F_{10}(1/3) \approx 5.5 > 5, \quad F_{16}(1/5) \approx 9.2 > 8.$$ 

[Editor’s note. Here are a few more details. For $b = 2$ and $a$ odd, one finds as in the proof of Proposition 3 that

$$F_6 \left( \frac{a}{2} \right) = 3F_2 \left( \frac{a}{2} \right) + 1 = 3F_2 \left( \frac{1}{2} \right) + 1 = F_6 \left( \frac{1}{2} \right),$$

and thus for $6 \leq n < 10$,

$$F_n \left( \frac{a}{2} \right) = F_6 \left( \frac{a}{2} \right) + \sum_{j=1}^{n} \left| \cos \frac{ja\pi}{4} \right| = F_6 \left( \frac{1}{2} \right) + \sum_{j=1}^{n} \left| \cos \frac{ja\pi}{4} \right| > 3 + (n - 6) \frac{1}{2} \quad \text{(as is easily checked)} \quad = \frac{n}{2}.$$ 

Similarly for $b = 3$, $a$ odd and not divisible by $3$, we use

$$F_{10} \left( \frac{a}{3} \right) = 3F_3 \left( \frac{a}{3} \right) + \frac{3}{2} = 3F_3 \left( \frac{1}{3} \right) + \frac{3}{2} = F_{10} \left( \frac{1}{3} \right)$$

to get, for $9 \leq n < 15$,

$$F_n \left( \frac{a}{3} \right) = F_{10} \left( \frac{a}{3} \right) + \sum_{j=1}^{n} \left| \cos \frac{j\alpha\pi}{6} \right| > 5 + \frac{n - 10}{2} = \frac{n}{2}.$$ 

For $b = 5$, $a$ odd and not divisible by $5$, we must use

$$F_{16} \left( \frac{a}{5} \right) \geq 3F_5 \left( \frac{a}{5} \right) + 1 + \cos \frac{2\pi}{5} = 3F_5 \left( \frac{1}{5} \right) + 1 + \cos \frac{2\pi}{5} = F_{16} \left( \frac{1}{5} \right)$$

to get, for $15 \leq n < 25$,

$$F_n \left( \frac{a}{5} \right) = F_{16} \left( \frac{a}{5} \right) + \sum_{j=17}^{n} \left| \cos \frac{j\alpha\pi}{10} \right| > 8 + \frac{n - 16}{2} = \frac{n}{2} \quad \Box$$

To finish, we need to verify that $F_3(x) \geq 1$, $F_5(x) \geq 2$ and $F_7(x) \geq 3$ for all $x$. It is possible to do this quite easily with the aid of a computer. By Lemma 1 it suffices to compute the values on the terms of the Farey sequence with odd numerator. The following tables give the results.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$F_3(x)$</th>
<th>$F_5(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>1.36+</td>
<td>2.65+</td>
</tr>
<tr>
<td>1/2</td>
<td>1.41+</td>
<td>2.39+</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2.73+</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$F_3(x)$</th>
<th>$F_5(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>1.76+</td>
<td>2.73+</td>
</tr>
<tr>
<td>1/4</td>
<td>2.93+</td>
<td>3.12+</td>
</tr>
<tr>
<td>1/3</td>
<td>3.12+</td>
<td>2.93+</td>
</tr>
<tr>
<td>1/2</td>
<td>3.12+</td>
<td>2.93+</td>
</tr>
<tr>
<td>2</td>
<td>2.65+</td>
<td>2.65+</td>
</tr>
</tbody>
</table>
No other solutions for this problem were received. The examples refuting the proposer’s conjecture for \( n = 4 \) and 6 were also found by RICHARD I. HESS, Rancho Palos Verdes, California; WALther J. NOuS, Ursulinen gymnasium, Innsbruck, Austria (\( n = 4 \) only); and CHARLTON WANG, student, Waterloo Collegiate Institute, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario. Hess gave the correct answer to the problem without proof.

The proposer also asks for the maximum value of

\[
|\sin \theta| + |\sin 2\theta| + \cdots + |\sin n\theta|
\]

for fixed \( n \). The editor is grateful he didn’t make this Crux 1542, part (b)!” However, readers may like to work on it.

Janous recalls a somewhat similar problem in a recent U.S.S.R. Olympiad, but has no more details. Maybe a reader can add more information?

* * * * *


Prove that for every positive integer \( n \) and every positive real \( x \),

\[
\sum_{k=1}^{n} \frac{x^{k^2}}{k} \geq x^{n(n+1)/2}.
\]

I. Solution by Kee-Wai Lau, Hong Kong.

We first show that for \( y > 0 \),

\[
f_n(y) := 1 + \frac{y^{n+2}}{n+1} - y^2 > 0.
\]

By differentiation it is easy to check that \( f_n(y) \) attains at

\[
y = \left( \frac{2(n+1)}{n+2} \right)^{1/n}
\]

its minimum value of

\[
1 + \frac{1}{n+1} \left( \frac{2(n+1)}{n+2} \right)^{1+\frac{2}{n}} - \left( \frac{2(n+1)}{n+2} \right)^{\frac{2}{n}} = 1 + \frac{1}{n+1} \left( \frac{2(n+1)}{n+2} \right)^{\frac{2}{n}} \left( \frac{2(n+1)}{n+2} - (n+1) \right)
\]

\[
= 1 - \frac{n}{n+2} \left( \frac{2(n+1)}{n+2} \right)^{\frac{2}{n}}.
\]

This minimum value is positive because for \( n \geq 2 \),

\[
\left( \frac{n+2}{n} \right)^{n/2} = \left( 1 + \frac{1}{n/2} \right)^{n/2} \geq 2 > \frac{2(n+1)}{n+2}.
\]
We now prove the inequality of the problem by induction. For \( n = 1 \) the inequality reduces to an identity. Suppose that the inequality holds for \( n = m \geq 1 \). Then

\[
\sum_{k=1}^{m+1} \frac{x^k}{k} - x^{(m+1)(m+2)/2} \geq x^{m+1}/2 + \frac{x^{(m+1)^2}}{m+1} - x^{m(m+1)(m+2)/2} \\
= x^{m+1}/2 \left( 1 + \frac{x^{(m+1)(m+2)/2}}{m+1} - x^{m+1} \right) \\
= x^{m+1}/2 f_m(x^{(m+1)/2}) > 0.
\]

This completes the solution of the problem.

II. Solution by the proposer.

By the A.M.-G.M. inequality,

\[
\frac{\sum_{k=1}^{n} k x^k}{\sum_{k=1}^{n} k} \geq \left( x \sum_{k=1}^{n} k^3 \right)^1/\sum_{k=1}^{n} k.
\]

Since

\[
\sum_{k=1}^{n} k^3 = \left( \frac{n(n+1)}{2} \right)^2 = \left( \sum_{k=1}^{n} k \right)^2,
\]

we get

\[
\sum_{k=1}^{n} k x^k \geq \frac{n(n+1)}{2} x^{n(n+1)/2}.
\]

Finally,

\[
\sum_{k=1}^{n} x^k = \sum_{k=1}^{n} \int_{0}^{x} k x^{k-1} \, dx \geq \frac{n(n+1)}{2} \int_{0}^{x} x^{n(n+1)/2 - 1} \, dx = x^{n(n+1)/2}.
\]

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; NICOS DIAMANTIS, student, University of Patras, Greece; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALThER JANOUS, Ursulengymnasium, Innsbruck, Austria; WEIXUAN LI, University of Ottawa, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and JEAN-MARIE MONIER, Lyon, France.

Janous's solution was the same as the proposer's. Janous and Falkowitz gave the generalization (proved as in II): if \( \lambda_1, \ldots, \lambda_n, \mu_1, \cdots, \mu_n \geq 0 \) satisfy

\[
\sum_{k=1}^{n} \lambda_k \mu_k^2 = \left( \sum_{k=1}^{n} \lambda_k \mu_k \right)^2,
\]

then

\[
\sum_{k=1}^{n} \lambda_k x^\mu_k \geq x \sum_{k=1}^{n} \lambda_k \mu_k
\]

for all \( x \geq 0 \).

Let \( P \) be an interior point of a parallelogram \( ABCD \), such that \( \angle APB = 2\angle ADP \) and \( \angle DCP = 2\angle DAP \). Prove that \( AB = PB = PC \).

Solution by Dan Sokolowsky, Williamsburg, Virginia.

Let \( \angle ADP = \alpha \) and \( \angle DAP = \beta \), so that \( \angle BAP = 2\alpha \) and \( \angle DCP = 2\beta \). Draw \( PO \parallel AB, PO \) cutting \( AD \), with \( PO = AB = CD \). Then \( ABPO, CDOP \) are parallelograms, so \( \angle AOP = 2\alpha \) and \( \angle DOP = 2\beta \). Extend \( PO \) to \( Q \) with \( OQ = OA \). Then

\[
\angle OQA = \angle OAP = \alpha = \angle ADP,
\]

so \( A, P, D, Q \) lie on a circle \( \kappa \). Hence \( \angle DQP = \angle DAP = \beta \), so

\[
\angle ODP = \angle DOP - \angle DQO = 2\beta - \beta = \beta,
\]

and hence \( OD = OQ = OA \), which implies \( O \) is the center of \( \kappa \). Then

\[
AB = OP = OA = OD = PB = PC.
\]

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; K.R.S. SAstry, Addis Ababa, Ethiopia; BRUCE SHAWYER, Memorial University of Newfoundland; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer (whose solution was somewhat similar to Sokolowsky’s).

Hut and Sastry gave converses. Sastry also showed that the result still holds if \( P \) is outside the parallelogram but on the same side of the line \( AD \) as \( B \) and \( C \) are; this can be proved as above. Finally, Sastry noted that such a point \( P \) exists if and only if \( BC \leq 2AB \).


Let \( a_1, a_2 \) be given positive constants and define a sequence \( a_3, a_4, a_5, \ldots \) by

\[
a_n = \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}}, \quad n > 2.
\]

Show that \( \lim_{n \to \infty} a_n \) exists and find this limit.

I. Comment by the editor.

Three readers, Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Marcin E. Kuczma, Warszawa, Poland; and John Lindsey, Northern Illinois University, Dekalb, noticed that this problem is equivalent to problem E3388 of the American Math. Monthly (p. 428 of the May 1990 issue), via the transformation \( a_i = \sqrt{2}/x_i \). The editor will
therefore wait to see what the Monthly publishes before deciding which solution, if any, to feature in Crux. The list of solvers of Crux 1548 will also be given at that time. It shouldn’t be too long a wait, although the Monthly seems to be even further behind in their solutions than we are!

* * * * *


In quadrilateral $ABCD$, $E$ and $F$ are the midpoints of $AC$ and $BD$ respectively. $S$ is the intersection point of $AC$ and $BD$. $H$, $K$, $L$, $M$ are the midpoints of $AB$, $BC$, $CD$, $DA$ respectively. Point $G$ is such that $FSEG$ is a parallelogram. Show that lines $GH$, $GK$, $GL$, $GM$ divide $ABCD$ into four regions of equal area.

Solution by C. Feu strange-Hamoir, Brussels, Belgium.


Démontrons que l’aire de $GKCL$ vaut le quart de l’aire de $ABCD$.

Joignons $EK$, $EL$, $KL$. L’homothétie de centre $C$ et de rapport 2 applique $CKEL$ sur $CBAD$ (puisque $K, E, L$ sont les milieux respectifs de $BC, AC, DC$), d’où

$$\text{aire}(CKEL) = \frac{1}{4} \text{aire}(CBAD).$$

$GE \parallel KL \parallel BD$, d’où $\text{aire}(GKL) = \text{aire}(EKL)$,

$$\text{aire}(GKCL) = \text{aire}(GKL) + \text{aire}(KLC) = \text{aire}(EKL) + \text{aire}(KLC) = \text{aire}(EKCL) = \frac{1}{4} \text{aire}(ABCD).$$

On démontre de même que

$$\text{aire}(GHBK) = \text{aire}(GMAH) = \text{aire}(GLDM) = \frac{1}{4} \text{aire}(ABCD).$$

Also solved by JORDI DOU, Barcelona, Spain; Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; E. SZEKERES, Turramurra, Australia; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson; and the proposer.
The proposer found the problem in Journal de Mathématiques Élémentaires, October 1917, no. 8494.

* * * * * * *

Let $A = [-1, 1]$. Find all functions $f : A \rightarrow A$ such that

$$|xf(y) - yf(x)| \geq |x - y|$$

for all $x, y \in A$.

Solution by R. P. Sealy, Mount Allison University.
There are four such functions:

$$f_1(x) = 1 \text{ for all } x \in A, \quad f_2(x) = -1 \text{ for all } x \in A,$$

$$f_3(x) = \begin{cases} 1 & \text{for } x \neq 0, \\ -1 & \text{for } x = 0, \end{cases} \quad f_4(x) = \begin{cases} -1 & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

That these four functions satisfy the condition is easily checked. We show that there are no others.

Letting $x = 0$ and $y = 1$, we see that $|f(0)| = 1$. Letting $x = -y \neq 0$, we get $|f(-x) + f(x)| \geq 2$, which implies (since $|f(x)| \leq 1$) that $f(-x) = f(x) = \pm 1$. Hence $|f(x)| = 1$ for all $x \in A$. Suppose there exists $x, y \in A$ satisfying $xy \neq 0$ and $f(x)f(y) = -1$. Then (since $f(-x) = f(x)$) there exists $x, y \in A$ satisfying $xy < 0$ and $f(x)f(y) = -1$. Then $|x + y| \geq |x - y|$, which is impossible for $xy < 0$.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; NICOS D. DIAMANTIS, student, University of Patras, Greece; MATHEW ENGLANDER, student, University of Waterloo; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALther JANOUS, Ursulinengymnasium, Innsbruck, Austria; JEAN-MARIE MONIER, Lyon, France; M. PARMENTER, Memorial University of Newfoundland; and the proposer.

* * * * * * *

Find a triangle $ABC$ with a point $D$ on $AB$ such that the lengths of $AB, BC, CA$ and $CD$ are all integers and $AD : DB = 9 : 7$, or prove that no such triangle exists.

1. Solution by Hayo Ahlburg, Benidorm, Spain.

Let $AD = kx$, $DB = \ell x$, $AD : DB = k : \ell$ (given), $AB = (k + \ell)x$, $BC = y$, $CA = z$, $CD = t$. These lengths are related by Stewart’s Theorem [1]

$$(k + \ell)(t^2 + k\ell x^2) = ky^2 + \ell z^2,$$

in our case ($k = 9$, $\ell = 7$)

$$16t^2 + 108x^2 = 9y^2 + 7z^2. \quad (1)$$
While the general case of such a quaternion quadratic equation is rather involved [2], I note that for the special case of isosceles triangles \( y = z \) (1) can be written simply as

\[
63x^2 = y^2 - t^2.
\]

With \( x = 1 \) (i.e., \( AD = 9 \) and \( DB = 7 \)) we find the two solutions

\[
\begin{array}{cccccc}
    y + t & y - t & AB & y = BC = CA & t = CD \\
    21 & 3 & 16 & 12 & 9 \\
    63 & 1 & 16 & 32 & 31.
\end{array}
\]

With \( x > 1 \), there can be many ways to split the product \( 63x^2 \) into two factors, leading to more solutions.

References:

Consider the situation that \( \triangle ABC \) is similar to \( \triangle ACD \). Then

\[
\frac{AB}{AC} = \frac{BC}{CD} = \frac{AC}{AD}.
\]

Write \( AD = 9f \), \( DB = 7f \), \( CD = x \), \( BC = a \), \( AC = b \), where \( f \) is an integer determining the scale. Then

\[
\frac{16f}{b} = \frac{a}{x} = \frac{b}{9f},
\]

or

\[
b = 12f \quad \text{and} \quad \frac{x}{a} = \frac{3}{4}.
\]

Points \( C \) lie on a circle with \( A \) as centre and radius \( b = 12f \). With \( g \) an integer in the range \( \{1, 2, \ldots, 7\} \), we have

\[
a = 4gf, \quad b = 12f, \quad c = 16f, \quad x = 3gf.
\]

On the boundaries \( (g = 1 \text{ and } g = 7) \) \( \triangle ABC \) is flat.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALther JANOUS, Ursulinen Gymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; JEAN-
MARIE MONIER, Lyon, France; D.J. SMEENK, Zaltbommel, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; JOSÉ YUSTY PITA, Madrid, Spain; and the proposer.

Many different examples were given. Festrachts-Hamoir and Smeenk came up with Ahlburg’s second isosceles triangle (in Solution I); incidentally the case $g = 3$ of Penning’s solution yields the first. The right-angled triangle corresponding to $g = 5$ of Penning’s solution was also found by Hut and Yusty.

$\ast\ast\ast\ast\ast\ast\ast$

$1552^\ast$. [1990: 171] Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

For each integer $n \geq 2$ let

$$x_n = \left(\frac{1}{2}\right)^n + \left(\frac{2}{3}\right)^n + \cdots + \left(\frac{n-1}{n}\right)^n.$$ 

Does the sequence $\{x_n/n\}$, $n = 2, 3, \ldots$, converge?

Solution by Richard Katz, California State University, Los Angeles.

Yes. In fact,

$$\lim_{n \to \infty} \frac{x_n}{n} = \int_0^1 e^{-1/x} \, dx.$$ 

To prove this, let

$$\underline{L} = \liminf_{n \to \infty} \frac{x_n}{n}, \quad \overline{L} = \limsup_{n \to \infty} \frac{x_n}{n}.$$ 

We will show that

(i) $\overline{L} \leq \int_0^1 e^{-1/x} \, dx$, and

(ii) for $0 < \delta < 1$ and $0 < \epsilon < e^{-1}$, $\int_\delta^1 (e^{-1} - \epsilon)^{1/x} \, dx \leq \frac{1}{\epsilon}$.

Clearly (i) and (ii) imply that

$$\underline{L} = \overline{L} = \lim_{n \to \infty} \frac{x_n}{n} = \int_0^1 e^{-1/x} \, dx.$$ 

For (i), note that (via the Riemann sum)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{-n/j} = \int_0^1 e^{-1/x} \, dx.$$ 

Now, $[(j-1)/j]^j$ increases to $e^{-1}$ as $j \to \infty$, so

$$\left(\frac{j-1}{j}\right)^j < e^{-1} \quad \text{for all } j.$$ 

Hence

$$x_n = \sum_{j=1}^{n} \left(\frac{j-1}{j}\right)^n < \sum_{j=1}^{n} e^{-n/j} \quad \text{for all } n.$$
and thus
\[ \limsup_{n \to \infty} \frac{x_n}{n} \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{-n/j} = \int_{0}^{1} e^{-1/x} \, dx. \]

Thus (i) is proved.

To prove (ii), given \( 0 < \delta < 1 \) and \( 0 < \epsilon < e^{-1} \), choose \( N \) such that
\[ \left( \frac{j-1}{j} \right)^j > e^{-1} - \epsilon \quad \text{for all } j \geq N. \]

Then for \( n \) large enough so that \( N/n < \delta \), we have
\[ \frac{1}{n} \sum_{j=[\delta n]}^{n} \left( e^{-1} - \epsilon \right)^{n/j} \leq \frac{1}{n} \sum_{j=1}^{n} \left( \frac{j-1}{j} \right)^n = x_n. \]

Taking the lim inf gives (ii). [Since \( \delta - 1/n < [\delta n]/n \leq \delta \), the lim inf of the left side is, via Riemann sums, the integral in (ii).]

The method given above can be used to prove the following slightly more general result. Let \( 0 \leq a_j \leq 1 \) for \( j = 1, 2, \ldots \), and suppose \( (a_j)^j \to a \). Put
\[ x_n = \sum_{j=1}^{n} (a_j)^n \]
for each \( n \). Then
\[ \lim_{n \to \infty} \frac{x_n}{n} = \int_{0}^{1} a^{1/x} \, dx. \]

Also solved by H.L. ABBOTT, University of Alberta; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; G.P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; and JOHN H. LINDSEY, University of Calgary. A partial solution from Wolfgang Gmeiner was sent in by the proposer. There was also one incorrect solution submitted.

All solvers in fact found the limit. Falkowitz also gave a generalization, although a weaker one than that of Katz. Abbott wonders whether \( x_n/n \) increases (if so this would instantly prove that \( \lim_{n \to \infty} (x_n/n) \) exists), and whether \( x_{n+m} \leq x_n + x_m \) holds for all \( n, m \geq 2 \).
Crux Mathematicorum

Volume 17, Number 8          October 1991

CONTENTS

The Olympiad Corner: No. 128                       R.E. Woodrow 225

Book Review                                        Andy Liu 235

Problems: 1671–1680                                 236

Solutions: 1448, 1553–1563, 1565, 1566             238
THE OLYMPIAD CORNER

No. 128

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Since we still haven’t heard from our IMO team representatives about the events in Sweden, I am putting off until the next issue a discussion of this year’s contest. Anyway, we still have some of the problems from last year’s IMO in China that were proposed to the jury, but not used. Once again I would like to thank Andy Liu, of the University of Alberta, who sent them to me. I also must correct a mistake I made in the last issue. Andy was an observer at the marking sessions in China but not a trainer of the team. I had remembered that he has helped to train the Hong Kong team in the past, and confused the rest. My apologies.

UNUSED PROBLEMS FROM THE 31ST IMO

1. Proposed by Hungary.

The incentre of the triangle ABC is K. The midpoint of AB is C_1 and that of AC is B_1. The lines C_1K and AC meet at B_2, the lines B_1K and AB at C_2. If the areas of the triangles AB_2C_2 and ABC are equal, what is the measure of \( \angle CAB \)?

2. Proposed by Ireland.

An eccentric mathematician has a ladder with n rungs which he always ascends and descends in the following ways: when he ascends each step, he covers a rungs, and when he descends each step, he covers b rungs, where \( a \) and \( b \) are fixed positive integers. By a sequence of ascending and descending steps, he can climb from ground level to the top rung of the ladder and come back down to ground level again. Find, with proof, the minimum value of \( n \) expressed in terms of \( a \) and \( b \).


Let \( a \) and \( b \) be integers with \( 1 \leq a \leq b \), and \( M = \lfloor (a + b)/2 \rfloor \). Define the function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) by

\[
f(n) = \begin{cases} 
  n + a & \text{if } n < M, \\
  n - b & \text{if } n \geq M.
\end{cases}
\]

Let \( f_1(n) = f(n) \), and \( f^{i+1}(n) = f(f^i(n)) \), for \( i = 1, 2, \ldots \). Find the smallest positive integer \( k \) such that \( f^k(0) = 0 \).

4. Proposed by Poland.

Let \( P \) be a point inside a regular tetrahedron \( T \) of unit volume. The four planes passing through \( P \) and parallel to the faces of \( T \) partition \( T \) into 14 pieces. Let \( f(P) \) be the total volume of those pieces which are neither a tetrahedron nor a parallelepiped. Find the exact bounds for \( f(P) \) as \( P \) varies inside \( T \).
5. Proposed by Poland.
Prove that every integer $k > 1$ has a multiple which is less than $k^4$ that can be written in the decimal system with at most four different digits.

6. Proposed by Romania.
Let $n$ be a composite positive integer and $p$ be a proper divisor of $n$. Find the binary representation of the smallest positive integer $N$ such that
\[
\frac{(1 + 2^p + 2^{n-p})N - 1}{2^n}
\]
is an integer.

7. Proposed by Romania.
Ten localities are served by two international airlines such that there exists a direct service (without stops) between any two of these localities, and all airline schedules are both ways. Prove that at least one of the airlines can offer two disjoint round trips each containing an odd number of landings.

8. Proposed by Thailand.
Let $a$, $b$, $c$ and $d$ be non-negative real numbers such that $ab + bc + cd + da = 1$. Show that
\[
\frac{a^3}{b + c + d} + \frac{b^3}{c + d + a} + \frac{c^3}{d + a + b} + \frac{d^3}{a + b + c} \geq \frac{1}{3}.
\]

9. Proposed by the U.S.A.
Let $P$ be a cubic polynomial with rational coefficients, and let $q_1, q_2, q_3, \ldots$ be a sequence of rational numbers such that $q_n = P(q_{n+1})$ for all $n \geq 1$. Prove that there exists $k \geq 1$ such that for all $n \geq 1$, $q_{n+k} = q_n$.

10. Proposed by the U.S.S.R.
Find all positive integers $n$ for which every positive integer whose decimal representation has $n - 1$ digits $1$ and one digit $7$ is prime.

11. Proposed by the U.S.S.R.
Prove that on a coordinate plane it is impossible to draw a closed broken line such that

(1) the coordinates of each vertex are rational;
(2) the length of each edge is $1$; and
(3) the line has an odd number of vertices.

Repeating ourselves? L.J. Upton, of Mississauga, Ontario writes pointing out that problem 2 [1991: 68-9] was previously discussed in *Eureka* (the original name of *Crux Mathematicorum*), in an article by T.M. Apostol [1977; 242-44].

Now we turn to the “archive problems”.
Each of the 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the six segments connecting them are all red.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.
A 4-point set as required will be called a red tetrahedron. We consider two cases:
Case 1. There is a point that is on at least four blue edges. Consider the four points at the opposite ends of four blue edges emanating from this point. Since there are no blue triangles these four points constitute a red tetrahedron.

Case 2. Every point is on at most three blue edges, hence on at least five red ones. So there are at least 45 red half-edges. But there are an even number of red half-edges, hence there are at least 46. Then there is a point $A$ that is on at least 6 red edges. Let $S$ be the set of 6 points at the opposite ends of 6 red edges from $A$. As is well known, there are 3 points of $S$ that span a monochromatic triangle, and since there are no blue triangles it must be red. These three points together with $A$ constitute a red tetrahedron.

Find all polynomials $f(x)$ with real coefficients such that
$$f(x) \cdot f(x + 1) = f(x^2 + x + 1).$$

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.

Substituting $x - 1$ for $x$ in the original equation
$$f(x)f(x + 1) = f(x^2 + x + 1)$$
we get
$$f(x - 1) \cdot f(x) = f(x^2 - x + 1).$$

There are two cases.
Case 1. If $f(x)$ is a constant polynomial we have $f(x) \equiv 0$ or $f(x) \equiv 1$.

Case 2. Suppose $f(x)$ is not a constant. Then $f(x)$ has at least one (complex) root. Let $a$ be a root with maximum absolute value. By (1) and (2), $f(a) = 0$ implies $f(a^2 + a + 1) = 0$, and $f(a^2 - a + 1) = 0$. Thus $a \neq 0$. If $a^2 + 1 \neq 0$, then $a, a^2 + a + 1, a^2 - a + 1, -a$ are the vertices of a parallelogram, and $|a^2 + a + 1|$ or $|a^2 - a + 1|$ is greater than $|a|$, contradicting its choice. So $a = \pm i$, and since $f$ has real coefficients both $i$ and $-i$ are roots of $f(x)$ and $f(x) = (x^2 + 1)^mg(x)$ where $m$ is a positive integer, and $g(x)$ is a polynomial which has real coefficients and is not divisible by $x^2 + 1$. By (1)
\[(x^2 + 1)^mg(x) \cdot (x^2 + 2x + 2)^mg(x + 1) = (x^4 + 2x^3 + 3x^2 + 2x + 2)^mg(x^2 + x + 1).\]

Now
\[(x^2 + 1)(x^2 + 2x + 2) = x^4 + 2x^3 + 3x^2 + 2x + 2. \tag{3}\]

This gives that
\[g(x) \cdot g(x + 1) = g(x^2 + x + 1),\]

i.e. \(g(x)\) satisfies the same functional equation as \(f(x)\). By the argument at the beginning of this case, \(g(x)\) (not being divisible by \(x^2 + 1\)) must be a constant polynomial, and hence, by Case 1, \(g(x) \equiv 1\). Thus if \(f(x)\) is non-constant and satisfies (1) we must have that
\[f(x) = (x^2 + 1)^m.\]

On the other hand (3) shows that (1) is then satisfied.

\[\star\]

   Let \(O\) be the centre of the circle through the points \(A, B, C,\) and let \(D\) be the midpoint of \(AB\). Let \(E\) be the centroid of the triangle \(ACD\). Prove that the line \(CD\) is perpendicular to the line \(OE\) if and only if \(AB = AC\).

   Set \(\overrightarrow{OA} = a, \overrightarrow{OB} = b, \overrightarrow{OC} = c\). Then
   \[\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OD} = \frac{3}{2} \overrightarrow{OA} + \frac{1}{2} \overrightarrow{OB} + \overrightarrow{OC} = \frac{1}{6}(3a + b + 2c)\]
   and
   \[\overrightarrow{CD} = \frac{1}{2} (\overrightarrow{CA} + \overrightarrow{CB}) = \frac{1}{2} (\overrightarrow{OA} - \overrightarrow{OC} + \overrightarrow{OB} - \overrightarrow{OC}) = \frac{1}{2} (a + b - 2c).\]
   Hence \(CD\) is perpendicular to \(OE\) if and only if \((3a + b + 2c, a + b - 2c) = 0\). Using the fact that \((a, a) = (b, b) = (c, c),\) this is equivalent to \((a, b - c) = (a, b) - (a, c) = 0.\) This just the condition that \(OA \perp CB,\) or that \(AB = AC.\)

   [Editor's note. One direction of this result was discussed in [1991: 105] as a solution to problem 1 of the 1983 British Mathematical Olympiad.]

   1985 people take part in an international meeting. In any group of three there are at least two individuals who speak the same language. If each person speaks at most five languages, then prove that there are at least 200 people who speak the same language.

   Solution by John Morvay, Springfield, Missouri.
   The assertion is surely true if some participant has a common language with the other 1984, since 1984/5 > 200. Thus we assume that some pair \(\{P_1, P_2\}\) have no common language. This pair forms 1983 triads with the remaining participants, each of which must
have a common language with $P_1$ or $P_2$ (or both). It follows that one of the pair, say $P_1$, has a common language with each of at least 992 participants. Since $P_1$ only speaks at most five languages, some one of them is spoken by at least 199 of the 992 people. Then that language is spoken by at least $199 + 1 = 200$ people, including $P_1$.

We now give solutions for some of the problems proposed but not used on the 1987 IMO in Havana, Cuba. These were given in the October and November 1987 numbers of the Corner.

**Finland 1.** [1987: 246]
In a Cartesian coordinate system, the circle $C_1$ has center $O_1 = (-2, 0)$ and radius 3. Denote the point $(1, 0)$ by $A$ and the origin by $O$. Prove that there is a positive constant $c$ such that for any point $X$ which is exterior to $C_1$,

$$|OX| - 1 \geq c \min\{|AX, AX'\}.$$ 

Find the smallest possible $c$.

*Solution by George Evangelopoulos, Athens, Greece.*

Denote by $D_1$ and $D_2$ the disks bounded by $C_1$ and the circle $C_2$ with center $A$ and radius 1. Clearly, $\min\{|AX, AX'\} = AX$ if $X \notin D_2$ and $\min\{|AX, AX'\} = AX'$ if $X \in D_2$.

If $X \notin D_1 \cup D_2$, set $t = \frac{|OX|}{AX}$. Then $X$ lies on the Apollonius circle $S_t$. On $S_t$, $(|OX| - 1)/AX = t - 1/AX$ is minimized when $AX$ is minimal; this is clearly the case when $X$ is on the boundary of $D_1 \cup D_2$. If $X$ is on $C_2$, $AX = 1$ and $t - 1/AX = t - 1$ is minimized when $X$ is as close to $O$ as possible; this means that $X$ is the intersection $X_0$ of $C_1$ and $C_2$. By some elementary trigonometry, $t = \frac{|OX_0|}{\sqrt{5}/3}$. If $X \in C_1$, one calculates that

$$t - 1/AX = \frac{\sqrt{1 + 24 \sin^2(\omega/2)} - 1}{6 \sin(\omega/2)},$$

where $\omega$ is the angle $XO_1A$. This is an increasing function of $\omega$. So even here $t - 1/AX$ is minimized at $X_0$.

If $X \in D_2 \setminus D_1$, we again consider $S_t$ such that $X \notin S_t$. On $S_t$, $(|OX| - 1)/AX^2 = t/AX - 1/AX^2$. This function of $AX$ assumes its minimum either when $AX$ takes its largest value or when it takes its smallest value, i.e., either on the boundary $C_2$ or the $x$-axis, where it reduces in either case to $t - 1$ and is minimized at $X_0$, or on the boundary $C_1$, where its expression is

$$\sqrt{1 + 24 \sin^2(\omega/2)} - 1 \cdot \frac{1}{[6 \sin(\omega/2)]^2}.$$ 

This decreases with $\omega$ and is minimized at $X_0$. So one can choose $c = \sqrt{5}/3 - 1$. It is also the smallest possible value of $c$. 

* * *
Poland 2. [1987: 248]

Let $P$, $Q$, $R$ be polynomials with real coefficients, satisfying $P^4 + Q^4 = R^2$. Prove that there exist real numbers $p$, $q$, $r$ and a polynomial $S$ such that $P = pS$, $Q = qS$ and $R = rS^2$.

Solution by George Evagelopoulos, Athens, Greece.

We prove by induction on $h(P, Q, R) = \deg P^4 + \deg Q^4 + \deg R^2$ that the conclusion holds for

$$\xi P^4 + Q^4 = R^2, \quad \text{with } \xi \in \{-1, +1\}. \quad (1)$$

The case $h(P, Q, R) = 0$ is obvious. Let us assume $h(P, Q, R) > 0$.

Suppose that an irreducible polynomial $F$ divides two of the polynomials $P^4$, $Q^4$, and $R^2$. Then $F$ divides the third one, and the uniqueness of factorization implies $F|P$, $F|Q$ and $F^2|R$, which readily completes the induction step.

Now suppose that $P$, $Q$, $R$ are pairwise co-prime. Then $-\xi P^4 = (Q^2 - R)(Q^2 + R)$ where $Q^2 - R$ and $Q^2 + R$ are co-prime. Hence, by the uniqueness of factorization, there are polynomials $A$, $B$ such that

$$Q^2 - R = \xi_1 A^4, \quad Q^2 + R = \xi_2 B^4 \quad (2)$$

where $\xi_1, \xi_2 \in \{-1, +1\}$. Adding up these two equations we get

$$\xi_1 A^4 + \xi_2 B^4 = C^2, \quad \text{where } C = \sqrt{2}Q.$$

Note that $\xi_1 = 1$ or $\xi_2 = 1$; otherwise $A = B = C = 0$, and then, by (1) and (2) $P = Q = R = 0$. Furthermore $A^4$, $B^4$ and $C^2$ are pairwise co-prime, because $A^4$ is co-prime with $B^4$. Finally $h(A, B, C) < h(P, Q, R)$ because

$$h(A, B, C) = \deg A^4 + \deg B^4 + \deg C^2 = \deg P^4 + \deg C^2$$

$$= \deg P^4 + \frac{1}{2} \deg Q^4 = h(P, Q, R) - \frac{1}{2} \deg Q^4 - \deg R^2,$$

and if $Q$ and $R$ were constant, so would be $P$. Now, the induction hypothesis implies that $A$, $B$, $C$ are constant. Hence, by (1) and (2), also $P$, $Q$ and $R$ are constant.

Poland 1. [1987: 278]

Let $F$ be a one-to-one mapping of the plane into itself which maps closed rectangles into closed rectangles. Show that $F$ maps squares into squares. Continuity of $F$ is not assumed.

Solution by George Evagelopoulos, Athens, Greece.

We consider an arbitrary rectangle $ABCD$. Let $O$ be the center of the rectangle, and $X$, $Y$, $Z$, $T$ the midpoints of the sides $AB$, $BC$, $CD$, $DA$ respectively. Let $P$, $P_{AB}$, $P_{BC}$, $P_{CD}$, $P_{DA}$ denote the rectangles $ABCD$, $ABYT$, $BCZX$, $CDTY$, $DAXZ$, and $a$, $b$ the segments $YT$, $XZ$, respectively. Thus we have

$$P = P_{AB} \cup P_{CD} = P_{BC} \cup P_{DA}, \quad a = P_{AB} \cap P_{CD}, \quad b = P_{BC} \cap P_{DA}.$$
We denote by \( Q, Q_{AB}, Q_{BC}, Q_{CD}, Q_{DA}, a', b' \) the respective images, i.e. \( Q = F(P) \), \( Q_{AB} = F(P_{AB}) \), etc. We have
\[
(1) \quad Q = Q_{AB} \cup Q_{CD}, \quad a' = Q_{AB} \cap Q_{CD}.
\]
This implies that
\[
(2) \quad Q_{AB} \text{ and } Q_{CD} \text{ are rectangles such that for some two parallel sides of } Q \text{ one of them is a side of } Q_{AB} \text{ and another is a side of } Q_{CD}.
\]
This in turn implies that
\[
(3a) \quad a' \text{ is a line segment whose endpoints lie on sides of the rectangle } Q \text{ and which is parallel to its two sides, or}
\]
\[
(3b) \quad a' \text{ is a rectangle whose vertices lie on sides of } Q \text{ and whose sides are parallel to the sides of } Q.
\]
From (3a) and (3b), and from similar conditions for the set \( b' \), it follows that
\[
(4) \quad \text{the set } a' \cap b' \text{ consists of a single point iff the sets } a' \text{ and } b' \text{ are perpendicular line segments which are parallel to sides of the rectangle } Q \text{ and whose endpoints lie on sides of } Q.
\]
Since \( F(O) = a' \cap b' \), the set \( a' \cap b' \) is a single point. From this, from the definition of \( a, b, a', b' \), and from (4) we obtain the following

**Lemma:** \( F \) maps line segments into line segments, sides of rectangles into sides of rectangles, vertices of rectangles into vertices of rectangles, and perpendicular line segments into perpendicular line segments.

From the lemma it follows that if \( ABCD \) is a square, then \( F(A)F(B)F(C)F(D) \) is its image and it is a rectangle. \( F \) maps diagonals of \( ABCD \) onto diagonals of its image. Since \( AC \) and \( BD \) are perpendicular, their images \( F(A)F(C) \) and \( F(B)F(D) \) are perpendicular, too. But this is possible if and only if \( F(A)F(B)F(C)F(D) \) is a square.

\* \* \*

We now return to solutions for problems from the December 1989 number of the Corner. We discuss the first four problems of the 24th Spanish Mathematics Olympiad [1989: 291].

1. Let \( \{x_n\}, n \in \mathbb{N} \), be a sequence of integers such that \( x_1 = 1, x_n < x_{n+1} \) for all \( n \geq 1 \) and \( x_{n+1} \leq 2n \) for all \( n \geq 1 \). Show that for each positive integer \( k \) there exist two terms \( x_r, x_s \) of the sequence such that \( x_r - x_s = k \).

**Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.**

Let \( k \) be a positive integer. Partition \( \{1, 2, 3, \ldots, 2k\} \) into \( k \) pairs \( \{1, k + 1\}, \{2, k + 2\}, \{3, k + 3\}, \ldots, \{k, 2k\} \). Since \( 1 = x_1 < x_2 < \ldots < x_{k+1} \leq 2k \), by the pigeon hole principle some 2 members of \( \{x_1, x_2, \ldots, x_{k+1}\} \) must comprise one of the pairs. The result follows.

2. We choose \( n \) points \((n > 3)\) on a circle, numbered from 1 to \( n \) in any order. We say that two non-adjacent points \( A \) and \( B \) are related if, in one of the arcs with \( A \) and \( B \) as endpoints, all the points are marked with numbers smaller than those of \( A \) and \( B \). Show that the number of pairs of related points is exactly \( n - 3 \).
Editor’s comment. Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, points out that this problem is identical to problem 3 of the 1987 Arany Daniel Competition [1989: 5]. A solution was submitted by Curtis Cooper, Central Missouri State University, Warrensburg, but we have already given a solution for the earlier occurrence. See [1990: 264–265].

3. Show that \(25x + 3y\) and \(3x + 7y\) are multiples of 41 for the same integer values of \(x\) and \(y\).

Editor’s comment. Several readers pointed out that the problem as stated can not be correct. These included Richard A. Gibbs, Fort Lewis College; Richard K. Guy, University of Calgary; Stewart Metchette, Culver City, CA; Bob Prielipp, University of Wisconsin–Oshkosh; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. Guy points out that \(25x + 3y\) is a multiple of 41 just if \(x \equiv 13y \mod 41\) (or \(y \equiv 19x \mod 41\)), while \(3x + 7y\) is a multiple of 41 just if \(x \equiv 25y \) or \(y \equiv 23x \mod 41\). The only solutions are \(x \equiv y \equiv 0 \mod 41\). Gibbs suggested that perhaps the 7 should be replaced by 2, since \(2(25x + 3y) = 41x + 3(3x + 2y)\). This gives \(25x + 3y\) is a multiple of 41 iff \(3x + 2y\) is, since 41 is prime. Bob Prielipp suggested that the correct statement may have been that \(25x + 3y\) and \(31x + 7y\) are multiples of 41 for the same values of \(x\) and \(y\). This is since \(25x + 3y \equiv 0 \mod 41\) iff \(16(25x + 3y) \equiv 0 \mod 41\) iff \(31x + 7y \equiv 0 \mod 41\), since \(16 \cdot 25 \equiv 31\) and \(16 \cdot 3 \equiv 7 \mod 41\).

4. The celebrated Fibonacci sequence is defined by

\[
a_1 = 1, \ a_2 = 2, \ a_i = a_{i-2} + a_{i-1} \ (i > 2).
\]

Express \(a_{2n}\) in terms of only \(a_{n-1}, a_n,\) and \(a_{n+1}\).

Solutions by Richard K. Guy, University of Calgary; O. Johnson, King Edward’s School, Birmingham, England; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang’s presentation.

We first show that for all \(n \geq 2\) and all \(k\) with \(2 \leq k < 2n - 1\)

\[
a_{2n} = a_k a_{2n-k} + a_{k-1} a_{2n-k-1}.
\]

When \(k = 2\) or \(2n - 2\), the right side of (*) equals \(a_2 a_{2n-2} + a_1 a_{2n-3} = 2a_{2n-2} + a_{2n-3} = a_{2n-2} + a_{2n-1} = a_{2n}\). Suppose that (*) holds for some \(k\) with \(2 \leq k < 2n - 2\). Then we have

\[
a_{2n} = a_k (a_{2n-k-1} + a_{2n-k-2}) + a_{k-1} a_{2n-k-1}
= (a_k + a_{k-1}) a_{2n-k-1} + a_k a_{2n-k-2}
= a_{k+1} a_{2n-(k+1)} + a_k a_{2n-k-2},
\]

completing the induction. Setting \(k = n\) in (*) we obtain

\[
a_{2n} = a_n^2 + a_{n-1}^2 = (a_n + a_{n-1})^2 - 2a_n a_{n-1}
= a_{n+1}^2 - 2a_n a_{n-1}.
\]
Remarks: (1) The expressions remain valid for $n = 1$ provided we set $a_0 = 1$.

(2) Clearly there are many other such expressions, e.g., using $a_{n-1} = a_{n+1} - a_n$ we have $a_{2n} = a_{n+1}^2 - 2a_{n+1}a_n + 2a_n^2$ and $a_{2n} = a_{n}^2 + a_{n-1}^2$ can be written as $a_{2n} = (a_{n+1} - a_{n-1})^2 + a_{n-1}^2 - 2a_{n+1}a_{n-1} + 2a_{n-1}^2$, etc.

We now turn to problems from the January 1990 number of Crux with solutions for the problems of the Singapore Mathematical Society Interschool Mathematical Competition, 1988 (Part B) [1990: 4–5].

1. Let $f(x)$ be a polynomial of degree $n$ such that $f(k) = \frac{k}{k+1}$ for each $k = 0, 1, 2, \ldots, n$. Find $f(n+1)$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea, and Murray S. Klamkin, University of Alberta.

Let $g(x) = (x+1)f(x) - x$. Then the given condition becomes $g(0) = g(1) = \cdots = g(n) = 0$. It follows that $g(x) = kx(x-1)\cdots(x-n)$ and

$$(x+1)f(x) = x + kx(x-1)\cdots(x-n).$$

Putting $x = -1$, we have $k = \frac{(-1)^{n+1}}{(n+1)!}$. We conclude that

$$f(x) = \frac{x + \frac{(-1)^{n+1}}{(n+1)!} x(x-1)\cdots(x-n)}{x+1}.$$ 

Thus

$$f(n+1) = \frac{n+1 + (-1)^{n+1}}{n+2} = \begin{cases} 1 & n \text{ odd} \\ \frac{n}{n+2} & n \text{ even}. \end{cases}$$

Editor’s note. Murray Klamkin points out that this problem has appeared previously, for example in M.S. Klamkin, USA Mathematical Olympiads 1972–1986, MAA, 1988, pp. 20–21. Slightly less elementary solutions were sent in by Duane M. Broline, Eastern Illinois University, Charleston; by Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

2. Suppose $\triangle ABC$ and $\triangle DEF$ in the figure are congruent. Prove that the perpendicular bisectors of $AD$, $BE$, and $CF$ intersect at the same point.

Comment by Duane M. Broline, Eastern Illinois University, Charleston.

The problem is obviously false as stated, as the accompanying diagram illustrates.

- $\ell$ is the $\perp$ bisector of $AD$.
- $m$ is the $\perp$ bisector of $BE$.
- $n$ is the $\perp$ bisector of $CF$. 
However, there is a result about perpendicular bisectors of congruent triangles:

Let $\Delta ABC$ be a triangle in the plane and $R$ any other point. If $\Delta DEF$ is the image of $\Delta ABC$ under any isometry which fixes $R$, then the perpendicular bisectors of $AD$, $BE$ and $CF$ intersect at $R$.

(This result follows since the perpendicular bisector of a line segment is the locus of all points which are equidistant from the two endpoints.)

3. Find all positive integers $n$ such that $P_n$ is divisible by 5, where $P_n = 1 + 2^n + 3^n + 4^n$. Justify your answer.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Stewart Metchette, Culver City, California; by Bob Prielipp, University of Wisconsin–Oshkosh; and by Michael Selby, University of Windsor.

Working mod 5, $P_n \equiv 1 + 2^n + (-2)^n + (-1)^n$. If $n$ is odd, $P_n \equiv 1 + 2^n - 2^n - 1 \equiv 0$ and so $5|P_n$. If $n = 2(2k + 1)$, $2^n = 4^{2k+1} \equiv -1$. So

$$P_n \equiv 1 + 2^n + 2^n + 1 \equiv 1 - 1 + 1 \equiv 0,$$

and $5|P_n$. Finally suppose $n = 4m$. Then $2^n \equiv (2^4)^m \equiv 1^m \equiv 1$ and $P_n \equiv 1 + 1 + 1 + 1 \equiv 4(\neq 0)$. Thus $P_n$ is divisible by 5 just in case $n$ is odd or twice an odd number.

Alternate Solution by Duane M. Broline, Eastern Illinois University, Charleston; by Murray S. Klamkin, University of Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

$P_n = 1 + 2^n + (5 - 2)^n + (5 - 1)^n$. If $n$ is odd, then $5|P_n$ by expansion of the last two terms using the binomial theorem. If $n$ is even, $n = 2m$, we get $P_n = 2(1 + 2^{2m}) + 5k$, by the same method. Now $1 + 2^{2m} = 1 + (5 - 1)^m$. For the latter to be divisible by 5, $m$ must be odd. Summarizing, $n$ must be odd or twice an odd number.

4. Prove that for any positive integer $n$, any set of $n + 1$ distinct integers chosen from the integers $1, 2, \ldots, 2n$ always contains 2 distinct integers such that one of them is a multiple of the other.

Editor’s note. Comments and solutions were received from Seung-Jin Bang, Seoul, Republic of Korea; Duane M. Broline, Eastern Illinois University, Charleston, Illinois; Murray S. Klamkin, University of Alberta; Bob Prielipp, University of Wisconsin–Oshkosh; and Michael Selby, University of Windsor.

This problem should be fairly well known. Murray Klamkin points out that with $n = 100$ it is given in D.O. Shklyarsky, N.N. Chentsov, I.M. Yaglom, Selected Problems and Theorems in Elementary Mathematics, Mir Publishers, Moscow, 1979. There the authors give two proofs, one by induction and the combinatorial proof below.

Solution. Let $x_1, x_2, \ldots, x_{n+1}$ be the $n + 1$ chosen integers. Then $x_i = 2^r m_i$ with $r_i \geq 0$ and $m_i$ odd for $1 \leq i \leq n + 1$. Since there are only $n$ odd numbers up to $2n$ we must have $m_i = m_j$ for some $i \neq j$. Then $x_i$ divides $x_j$ or $x_j$ divides $x_i$ according to whether $r_i < r_j$ or $r_i > r_j$. 


5. Find all positive integers \( x, y, z \) satisfying the equation \( 5(xy + yz + zx) = 4xyz \).

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Duane M. Broline, Eastern Illinois University, Charleston, Illinois; by Stewart Metcalf, Culver City, California; by Bob Prielipp, University of Wisconsin-Oshkosh; by Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. The write-up given is Wang’s.

The given equation can be rewritten as

\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{5} \tag{*}
\]

Without loss of generality, we may assume that \( 1 \leq x \leq y \leq z \). Since \( x \), \( y \) and \( z \) are positive \( x = 1 \) is clearly impossible. On the other hand, if \( x \geq 4 \), then

\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{4} < \frac{4}{5}.
\]

Thus (\( * \)) can have integer solutions only if \( x = 2 \) or \( x = 3 \).

When \( x = 3 \), (\( * \)) becomes \( \frac{1}{y} + \frac{1}{z} = \frac{2}{15} \). If \( y \geq 5 \), then \( \frac{1}{y} + \frac{1}{z} \leq \frac{2}{5} < \frac{7}{15} \). Thus \( y = 3 \) or 4. In either case, we can easily find that the corresponding value for \( z \) is not an integer.

When \( x = 2 \), (\( * \)) becomes \( \frac{1}{y} + \frac{1}{z} = \frac{3}{10} \). If \( y \leq 3 \), then \( \frac{1}{y} + \frac{1}{z} > \frac{1}{3} > \frac{3}{10} \). If \( y \geq 7 \) then \( \frac{1}{y} + \frac{1}{z} \leq \frac{3}{7} < \frac{3}{10} \). Thus \( y = 4, 5 \), or 6.

For \( y = 4 \), we solve and get \( z = 20 \), and a solution \((2, 4, 20)\). For \( y = 5 \), we have \( z = 10 \) and the solution \((2, 5, 10)\). For \( y = 6 \) we find \( z = 15/2 \), which is not an integer.

To summarize, the given equation has exactly 12 solutions obtained by permuting the entries of each of the two ordered triples \((2, 4, 20)\) and \((2, 5, 10)\).

\[
\ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast
\]

This completes the Corner for this month. Send me your nice solutions!

\[
\ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast
\]

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.


This may be considered a sequel to the author’s earlier problem anthology Mathematical Morsels. However, in the present volume, almost all of the 57 morsels appeared previously in Crux Mathematicorum. The author has foraged in its fertile soil before. See for instance his Mathematical Gems III, especially Chapter 7.

Let us examine first the material from outside Crux. Morsel 51 presents a new proof of a theorem of Moessner due to K. Post. Morsel 55 illustrates with an example the
“Probabilistic Methods in Combinatorics” discussed by P. Erdős and J. Spencer in their monograph of that title (the reference is inadvertently omitted). Morsel 56 is based on a student project by W. K. Chan on point sets not determining right triangles. The original papers were by A. Seidenberg and H. L. Abbott. Morsel 57 reexamines a morsel from the author’s earlier Mathematical Morsels.

This is excellent material. Its presence, however, also serves to keep “Crux Mathematicorum” off the cover of the book. Nevertheless, it is gratifying to see the labour of love by Léo Sauvé and Fred Maskell duly acknowledged in the Preface.

Practically all of the remaining 53 morsels are taken from the regular Problem Sections of Crux. (A reference to Crux 1119 [1987: 258] should have been made in Morsel 49, which is based on an outstanding expository article by S. Wagon.) The only other exceptions are Morsels 1, 26 and 46, which are taken from the Olympiad Corners in 1979. Why they are not included in the first of eight sections titled “Gleanings from Murray Klamkin’s Olympiad Corners, 1979-1986” is puzzling.

The author certainly has good taste in the choice of material. For instance, it would have been a grievous omission if Gregg Patruno’s brilliant proof of Archimedes’ “Broken-chord Theorem” had not been included. Happily, the readers are reacquainted with this gem as Morsel 8.

In the Preface, the author freely admits that great liberty has been taken with the work of the original contributors to Crux. The author’s tendency is to take apart an argument and analyse it step by step. Depending on the intended audience, this is a valuable service. However, there are times when the “down marketing” may have gone a bit too far. It would also seem desirable to have some sort of classification of the problems by subject matter, along the line of S. Rabinowitz’s ambitious project (see his “Letter to the Editor” on [1991: 96]).

The proposers and solvers of the problems in the regular Problem Sections are listed at the end of the book. There are only some minor glitches; for instance, the references for Morsels 5, 6 and 10 have been permuted, and in the reference to Morsel 29 the Hungarian journal KOMAL is called a Russian journal. On the other hand, there is no complete or consistent acknowledgement of the original contributors to the Olympiad Corners. This is particularly confusing since the author has included solutions to a small number of problems to which no solutions have yet appeared in Crux.

The book is certainly up to the author’s high standards, and is a good addition to the bookshelves, especially since the Canadian Mathematical Society has regretfully stopped putting out bound volumes of Crux.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.
Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1992, although solutions received after that date will also be considered until the time when a solution is published.

1671. Proposed by Toshio Seimiya, Kawasaki, Japan.
A right triangle $ABC$ with right angle at $A$ is inscribed in a circle. Let $M$, $N$ be the midpoints of $AB$, $AC$, and let $P$, $Q$ be the points of intersection of the line $MN$ with $BC$. Let $D$, $E$ be the points where $AB$, $AC$ are tangent to the incircle. Prove that $D$, $E$, $P$, $Q$ are concyclic.

1672. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.
Show that for positive real numbers $a, b, c, x, y, z$,
\[
\frac{a}{b+c}(y+z) + \frac{b}{c+a}(z+x) + \frac{c}{a+b}(x+y) \geq 3 \left( \frac{xy + yz + zx}{x+y+z} \right),
\]
and determine when equality holds.

Triangle $ABC$ is nonequilateral and has angle $\beta = 60^\circ$. $A'$ is an arbitrary point on line $BA$, not coinciding with $B$ or $A$. $C'$ is an arbitrary point of $BC$, not coinciding with $B$ or $C$.
(a) Show that the Euler lines of $\Delta ABC$ and $\Delta A'BC'$ are parallel or coinciding.
(b) In the case of coincidence, show that the circumcircles of all such triangles $A'BC'$ meet the circumcircle of $ABC$ at a fixed point.

1674. Proposed by Murray S. Klamkin, University of Alberta.
Given positive real numbers $r$, $s$ and an integer $n > r/s$, find positive $x_1, x_2, \ldots, x_n$ so as to minimize
\[
\left( \frac{1}{x_1^r} + \frac{1}{x_2^r} + \cdots + \frac{1}{x_n^r} \right) (1 + x_1)^s (1 + x_2)^s \cdots (1 + x_n)^s.
\]

Let $V_1, V_2, \ldots, V_n$ denote the vertices of a regular $n$-gon inscribed in a unit circle $C$ where $n \geq 3$, and let $P$ be an arbitrary point on $C$. It is known that $\sum_{k=1}^n PV_k^2$ is a constant.
(a) Show that $\sum_{k=1}^n PV_k^{-1}$ is also a constant.
(b) Does there exist a value of $m \neq 1, 2$ and a value of $n \geq 3$ such that $\sum_{k=1}^n PV_k^{2m}$ is independent of $P$?

$OA$ is a fixed radius and $OB$ a variable radius of a unit circle, such that $\angle AOB \leq 90^\circ$. $PQRS$ is a square inscribed in the sector $OAB$ so that $PQ$ lies along $OA$. Determine the minimum length of $OS$.


Evaluate (without rearranging)

$$1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \cdots.$$ 

1678. Proposed by George Tsintsifas, Thessaloniki, Greece.

Show that

$$\sqrt{\sqrt{a} + \sqrt{b} + \sqrt{c}} \leq \sqrt{2(r_a + r_b + r_c)},$$

where $a, b, c$ are the sides of a triangle, $s$ the semiperimeter, and $r_a, r_b, r_c$ the exradii.

1679. Proposed by Len Bos and Bill Sands, University of Calgary.

$A_1A_2A_3A_4$ is a unit square in the plane, with $A_1(0,1)$, $A_2(1,1)$, $A_3(1,0)$, $A_4(0,0)$. $A_5$ is the midpoint of $A_1A_2$, $A_6$ the midpoint of $A_2A_3$, $A_7$ the midpoint of $A_3A_4$, $A_8$ the midpoint of $A_4A_5$, and so on. This forms a spiral polygonal path $A_1A_2A_3A_4A_5A_6A_7A_8 \ldots$ converging to a unique point inside the square. Find the coordinates of this point.

1680. Proposed by Zun Shan and Ji Chen, Ningbo University, China.

If $m_a, m_b, m_c$ are the medians and $r_a, r_b, r_c$ the exradii of a triangle, prove that

$$\frac{r_br_c}{m_bm_c} + \frac{r_cr_a}{m_cm_a} + \frac{r_ar_b}{m_am_b} \geq 3.$$

* * * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.


If $A, B, C$ are the angles of a triangle, prove that

$$\frac{2}{3} \left( \sum \sin \frac{A}{2} \right)^2 \geq \sum \cos A,$$

with equality when $A = B = C$. 


Editor's note. The statement of this problem is false, but Murray Klamkin conjectured on [1990: 222] that it should be true for acute triangles. Three readers have since sent in proofs of this conjecture. Two of them contain different “best possible” results, so here they are.

II. Solution by G.P. Henderson, Campbellcroft, Ontario.
We choose the notation so that \( A \geq B \geq C \). Then we prove:
(a) if \( B < 2 \arcsin \left( \frac{\sqrt{3}-1}{2} \right) \approx 21.1^\circ \), the inequality is false;
(b) if \( 2 \arcsin \left( \frac{\sqrt{3}-1}{2} \right) \leq B < 30^\circ \), it is false for small values of \( C \) and becomes true as \( C \) increases toward \( B \);
(c) if \( B \geq 30^\circ \), the inequality is true.

For acute triangles, \( B \geq 45^\circ \). Therefore M.S. Klamkin’s conjecture that the inequality is true for such triangles is correct.

Set
\[
X = \sin(C/2), \quad v = \sin(B/2), \quad Y = \sin(A/2).
\]

Then
\[
0 < v < \sqrt{2}/2 \quad (1)
\]
and
\[
0 < X \leq v < Y < 1. \quad (2)
\]

Since \( \sum A/2 = 90^\circ \),
\[
X^2 + 2vXY + Y^2 = 1 - v^2. \quad (3)
\]

In terms of \( X, Y \) and \( v \) the proposed inequality is
\[
8X^2 + 4XY + 8Y^2 + 4vX + 4vY + 8v^2 - 9 \geq 0. \quad (4)
\]

Both (3) and (4) are simpler if we rotate the \( XY \)-axes through \( 45^\circ \) and change the scales. Set
\[
X = x - y, \quad Y = x + y.
\]
(2) and (3) become
\[
|x - v| \leq y < x \quad (5)
\]
and
\[
2(1 + v)x^2 + 2(1 - v)y^2 = 1 - v^2. \quad (6)
\]

The inequality is now
\[
20x^2 + 12y^2 + 8vx + 8v^2 - 9 \geq 0. \quad (7)
\]

Geometrically, for a given \( v \) satisfying (1), a certain arc of the ellipse (6) is to be outside the ellipse (7).

We use (6) to eliminate \( y \). From (5), \( y \geq 0 \). Therefore
\[
y = \sqrt{\frac{1 - v^2}{2(1 - v)} - \frac{2(1 + v)x^2}{2(1 - v)}}. \quad (8)
\]
The first part of (5) is equivalent to
\[(2x + 1 - v)(2x + 2v^2 - v - 1) \leq 0.\]
The first factor can be omitted because \(x > 0\) (from (5)) and \(v < 1\). From the second part of (5),
\[4x^2 > 1 - v^2.\]
Thus (5) is equivalent to
\[x_1 < x \leq x_2\]
where
\[x_1 = \frac{\sqrt{1 - v^2}}{2},\]
and
\[x_2 = \frac{1 + v - 2v^2}{2} = \frac{(1 - v)(1 + 2v)}{2}.\]
For these values of \(x\), the expression under the radical sign in (8) is positive. Using (8), (7) becomes
\[f(x) = 8(1 - 4v)x^2 + 8v(1 - v)x - 8v^3 + 2v^2 + 9v - 3 \geq 0.\]
We are to determine the values of \(x\) and \(v\) that satisfy (1), (9) and (10). We find
\[f(x_1) = (1 - v)(4v\sqrt{1 - v^2} - 1) = (1 - \sin \frac{B}{2})(2\sin B - 1),\]
\[f(x_2) = (1 - v)(32v^4 - 16v^3 - 12v^2 + 8v - 1)
\quad = \frac{1}{2}(1 - v)(2v - 1)^2(4v + 1 + \sqrt{3})(4v + 1 - \sqrt{3}).\]

Case (a): \(B < 2 \arcsin(\sqrt{3} - 1)/4\).
We have
\[0 < v < (\sqrt{3} - 1)/4 < 1/4, \quad f(x_1) < 0, \quad f(x_2) < 0,\]
and \(f\) is convex. Therefore (10) is false for \(x_1 < x \leq x_2\).

Case (b): \(2 \arcsin(\sqrt{3} - 1)/4 \leq B < 30^\circ\).
The sign of \(f\) changes from negative to positive as \(x\) increases from \(x_1\) to \(x_2\), that is, as \(C\) increases from 0 to \(B\). Therefore (10) is false for small values of \(C\) and becomes true as \(C\) approaches \(B\).

Case (c): \(B \geq 30^\circ\).
We have
\[\frac{1}{4} < \frac{\sqrt{6} - \sqrt{2}}{4} \leq v < \frac{\sqrt{2}}{2}, \quad f(x_1) \geq 0, \quad f(x_2) \geq 0,\]
and \(f\) is concave. Therefore (10) is true for \(x_1 < x \leq x_2\).
III. Solution by Marcin E. Kuczma, Warszawa, Poland.

While failing in its full generality, the inequality is indeed valid for acute triangles, as conjectured by M.S. Klamkin. Actually, we prove that the inequality holds in every triangle of angles not exceeding arccos((1 − 4\sqrt{3})/8) ≈ 137.8°; the bound cannot be improved upon.

Assume \( A \geq B \geq C \) and let

\[
\varphi = \frac{B + C}{2}, \quad \psi = \frac{B - C}{2}, \quad t = 2\cos \varphi - 1, \quad x = \cos(\psi/2).
\]

Then \( A = \pi - 2\varphi, \ B = \varphi + \psi, \ C = \varphi - \psi, \ \pi/3 \geq \varphi \geq \psi \geq 0, \ 0 \leq t \leq 1, \ 0 \leq x \leq 1, \ 
\cos \psi = 2x^2 - 1, \ \cos \varphi = (1 + t)/2, \) and we have

\[
\begin{align*}
\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} &= \sin \frac{\pi - 2\varphi}{2} + 2\sin \frac{\varphi}{2} \cos \frac{\psi}{2} \\
&= \cos \varphi + 2\sqrt{1 - \cos^2 \varphi} \cdot \cos \frac{\psi}{2} \\
&= \frac{1 + t}{2} + \sqrt{1 - t} x, \\
\cos A + \cos B + \cos C &= (1 - 2\cos^2 \varphi) + 2\cos \varphi \cos \psi \\
&= 1 - \frac{(1 + t)^2}{2} + (1 + t)(2x^2 - 1).
\end{align*}
\]

After small manipulation, the inequality under investigation takes on the form

\[
f(t, x) = 4(1 + 2t)x^2 - 2(1 + t)\sqrt{1 - t} x - (2 + 7t + 2t^2) \leq 0.
\]

For \( t, x \in [0, 1], \)

\[
f(t, 1) - f(t, x) = 2(1 - x)[2(1 + 2t)(1 + x) - (1 + t)\sqrt{1 - t}] \geq 0,
\]

with equality only for \( x = 1 \). So it suffices to examine \( f \) for \( x = 1 \). With some routine calculation we arrive at

\[
f(t, 1) \begin{cases} < 0 & \text{for } t \in (0, \sqrt{3}/2), \\ = 0 & \text{for } t = 0 \text{ and } t = \sqrt{3}/2, \\ > 0 & \text{for } t \in (\sqrt{3}/2, 1].
\end{cases}
\]

The “border value” \( t = \sqrt{3}/2 \) corresponds to

\[
\cos A = -\cos 2\varphi = 1 - 2\cos^2 \varphi = 1 - (1 + t)^2/2 = (1 - 4\sqrt{3})/8;
\]

clearly, \( t = 0 \) corresponds to \( A = \pi/3, \) and \( x = 1 \) to \( B = C. \n\)

Conclusion: writing \( \alpha = \arccos((1 - 4\sqrt{3})/8) \approx 137.8°, \)

if \( A \leq \alpha, \) the inequality is true for all \( B, C; \)

if \( A > \alpha, \) the inequality can fail to hold, and certainly does so when \( B = C. \)
For $A < \alpha$, equality requires $t = 0$ and $x = 1$, which occur when the triangle is equilateral.

Klamkin's conjecture was also proved by JOHN LINDSEY, Northern Illinois University, Dekalb.

The editor did not succeed in combining the above proofs. Maybe some reader can easily derive one result from the other. Note, by the way, that Kuczma's upper bound on $A$ and Henderson's lower bound on $B$ are related:

$$\arccos \left( \frac{1 - 4\sqrt{3}}{8} \right) = 180^\circ - 4 \arcsin \left( \frac{\sqrt{3} - 1}{4} \right),$$

corresponding to the isosceles triangle with

$$A = \arccos \left( \frac{1 - 4\sqrt{3}}{8} \right) \approx 137.8^\circ, \quad B = C = 2 \arcsin \left( \frac{\sqrt{3} - 1}{4} \right) \approx 21.1^\circ,$$

for which equality holds in the problem.

Proposed by Murray S. Klamkin, University of Alberta.

It has been shown by Oppenheim that if $ABCD$ is a tetrahedron of circumradius $R$, $a, b, c$ are the edges of face $ABC$, and $p, q, r$ are the edges $AD, BD, CD$, then

$$64R^4 \geq (a^2 + b^2 + c^2)(p^2 + q^2 + r^2).$$

Show more generally that, for $n$-dimensional simplexes,

$$(n + 1)^4R^4 \geq 4E_0E_1,$$

where $E_0$ is the sum of the squares of all the edges emanating from one of the vertices and $E_1$ is the sum of the squares of all the other edges.

1. Solution by Marcin E. Kuczma, Warszawa, Poland.

A slightly stronger estimate can be obtained, namely,

$$E_0E_1 \leq \left( \frac{4}{3n} \right)^3 R^4. \quad (1)$$

The right-hand expression in (1) is less than the claimed one $3(n + 1)^4R^4/4$, except for $n = 3$, when the two values are equal.

Let $v_0, \ldots, v_n$ be the vectors from the circumcenter to the vertices. Denote by $w$ the centroid of the system $(v_1, \ldots, v_n)$. So

$$|v_i| = R \quad (i = 0, 1, \ldots, n);$$

$$w = \frac{1}{n}(v_1 + \cdots + v_n), \quad |w| \leq R.$$
By definition,

\[ E_1 = \frac{1}{2} \sum_{i,j=1}^{n} |v_i - v_j|^2 = \sum_{i,j=1}^{n} (R^2 - v_i \cdot v_j) = n^2 R^2 - \left( \sum_{i=1}^{n} v_i \right)^2 = n^2 (R^2 - |w|^2); \]

\[ E_0 = \sum_{i=1}^{n} |v_0 - v_i|^2 = \sum_{i=1}^{n} (2R^2 - 2v_0 \cdot v_i) = 2n(R^2 - v_0 \cdot w); \]

dots denote inner (scalar) products of vectors. Since \( v_0 \cdot w \geq -R|w| \), and using the A.M.-G.M. inequality, we obtain

\[ E_0 E_1 = 2n^3(R^2 - v_0 \cdot w)(R^2 - |w|^2) \leq 2n^3(R^2 + R|w|)(R^2 - |w|^2) \]

\[ = 8n^3R \left( R - |w| \right)^{1/3} \left( \frac{R + |w|}{2} \right)^{2/3} \]

\[ \leq 8n^3R \left\{ \frac{1}{3} \left( R - |w| \right) + \frac{2}{3} \left( \frac{R + |w|}{2} \right) \right\}^3 = \left( \frac{4}{3} n \right)^3 R^4. \]

**Remark.** Assume \( n \geq 2 \). Then the estimate in (1) is sharp; it turns into equality if and only if \( 3w = -v_0 \). Note that, except for \( n = 2 \) and \( 3 \), there are many nonisometric optimal configurations and that, except for \( n = 3 \), the regular simplex is *not* among them.

II. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let \( O \) be the circumcenter and \( G \) the centroid of the simplex \( A_0 \cdots A_n \). Then the following identities are more or less familiar, at least for triangles and tetrahedra:

\[ \sum_{i=0}^{n} GA_i^2 = (n + 1)(R^2 - OG^2), \quad E_0 + E_1 = (n + 1) \sum_{i=0}^{n} GA_i^2 \]

(see, e.g., various simplex-problems by Murray Klamkin and/or George Tsintsifas in *Crux*, or Mitrović et al, *Recent Advances in Geometric Inequalities*, p. 493 and p. 502). Therefore

\[ E_0 + E_1 = (n + 1)^2(R^2 - OG^2) \]

and we get the better inequality

\[ 4E_0 E_1 \leq (E_0 + E_1)^2 = (n + 1)^4(R^2 - OG^2)^2. \]

Also solved by G.P. HENDERSON, Campbellecroft, Ontario; and the proposer.

With this “formidable four” (the three solvers plus the proposer) in action, it’s not surprising that all of them found stronger results! Henderson in fact gave the same improvement as Kuczma, with the same remarks at the end. In another direction, the proposer showed that \( W^4 R^4 \geq 4E_0 E_1 \) where

\[ E_0' = \sum_{j=1}^{n} w_0 w_j a_{0j}^2 \quad \text{and} \quad E_1' = \sum_{1 \leq i < j} w_i w_j a_{ij}^2 \]
are weighted sums, $W = \sum_{i=0}^{n} w_i$ the sum of the weights, and where $a_{ij}$ is the length of the edge between vertices $i$ and $j$. 

$\star \quad \star \quad \star \quad \star \quad \star \quad \star$


Describe all finite sets $S$ in the plane with the following property: if two straight lines, each of them passing through at least two points of $S$, intersect in $P$, then $P$ belongs to $S$.

Solution by Chris Wildhagen, Rotterdam, The Netherlands.

Suppose that a finite set $F$ of points in the plane has exactly one of the following properties:

I: the points of $F$ are collinear;

II: $|F| = 5$, and the points of $F$ are the vertices of a parallelogram together with the point of intersection of the diagonals;

III: all points of $F$, except one, are on a straight line.

Clearly such a set has the “closure property” as required by the problem. Conversely, each set $F$ obeying the conditions of the problem satisfies I, II or III as we shall show.

So take a finite set $F$ with the closure property. We may assume that $|F| \geq 4$, else $F$ satisfies I or III and we are done. If $A, B, C, D$ are 4 points of $F$, no 3 of which are collinear, then we can group them in two pairs, say $\{A, B\}$ and $\{C, D\}$, such that the two lines $AB$ and $CD$ intersect. This observation shows that $F$ contains 3 collinear points lying on a line $\ell$, say.

For each point $L \in F \cap \ell$, and each point $P$ of $F$ not on $\ell$, let $\theta(P, L)$ be the non-obtuse angle between $PL$ and $\ell$ (we assume that I doesn’t hold, else we are done). Choose $P$ and $L$ such that $\theta(P, L)$ is minimal and if there are several choices for $P$ choose the one with the distance $d(P, L)$ between $P$ and $L$ minimal.

Let $A$ and $B$ be two other points on $\ell$ belonging to $F$. Choose points $L'$ on line $PL$ and $B'$ on line $PB$, not necessarily in $F$, such that $P$ lies between $L$ and $L'$ and between $B$ and $B'$.

We claim that $F$ contains no point different from $P$ and on the same side of $\ell$ as $P$. This follows from the following six facts:

(i) $F$ contains no point in $(PLB) = \text{the interior of angle } PLB$ (by minimality of $\theta(P, L)$);

(ii) $F$ contains no point on the open half-line $\overline{PL'}$ (the line through $A$ and any point of $\overline{PL'}$ intersects $(PB) = \text{the interior of segment } PB \subseteq (PLB)$; now use (i));

(iii) $F$ contains no point in $(B'PL')$ (the line through $B$ and any point of $(B'PL')$ intersects $\overline{PL'}$; now use (ii));

(iv) $F$ contains no point on $(PL)$ (by minimality of $d(P, L)$);
(v) \( \mathcal{F} \) contains no point on \( \overline{PB'} \) (the line through \( A \) and any point of \( \overline{PB'} \) intersects \( (PL) \); now see (iv));

(vi) \( \mathcal{F} \) contains no point in \( (PBL) \) (the line through \( B \) and any point of \( (PBL) \) intersects \( (PL) \); now see (iv)).

If \( \mathcal{F} \) contains no point on that side of \( \ell \) which doesn’t contain \( P \), then by the claim we see that \( \mathcal{F} \) satisfies III. So suppose that \( \mathcal{F} \) does contain such a point; call it \( Q \). By the claim, \( Q \) is the only point of \( \mathcal{F} \) on that side. Now it’s easy to see that \( \ell \) contains exactly 3 points, else one can create a new point of \( \mathcal{F} \) on some side of \( \ell \). Moreover \( PQ \cap \ell = A \), \( QL \parallel PB \), \( QB \parallel PL \). Thus \( \mathcal{F} \) satisfies II.

Also solved by JORDI DOU, Barcelona, Spain; and the proposer. Another reader sent in the correct solution without proof. There was also one incorrect solution received.

Walther Janous recalls the problem in an article in Kvant “a long time ago”, and also a similar Monthly problem, but could not supply details.

* * * * *


\( ABCD \) is an isosceles trapezoid, with \( AD \parallel BC \), whose circumcircle has center \( O \).

Let \( PQRST \) be a rhombus whose vertices \( P, Q, R, S \) lie on \( AB, BC, CD, DA \) respectively.

Prove that \( Q, S \) and \( O \) are collinear.

Solution by Dan Sokolowsky, Williamsburg, Virginia.

Let \( M \) be the midpoint of \( AB \) and \( N \) the midpoint of \( CD \). It is then easily seen that \( MN \parallel BC \), and that \( MN \) bisects \( QS \), hence that \( MN \) passes through the common midpoint \( X \) of \( QS \) and \( PR \). Let \( RT \parallel MN \) (as shown). Then, since \( PX = XR \), \( PM = MT \). Clearly \( MT = RN \), so \( PM = RN \).

Also, since \( AB = CD \), \( OM = ON \), while \( \angle OMP = \angle ONR = 90^\circ \). Hence \( \angle OMP \cong \angle ONR \), so \( OP = OR \). Then, since \( XP = XR \), \( OX \) is the perpendicular bisector of \( PR \), as is \( QS \). Hence the lines \( OX \) and \( QS \) coincide, which implies that \( Q, S \) and \( O \) are collinear.

Also solved by JORDI DOU, Barcelona, Spain; L.J. HUT, Groningen, The Netherlands; WALThER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; JOHN RAUSEN, New York; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

* * * * *


Let \( \lambda \) and \( n \) be fixed positive integers, not both 1. Prove that the equation

\[
\frac{x^2 + y^2}{\lambda xy + 1} = n^2
\]

has infinitely many natural number solutions \((x, y)\).
Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

Multiplying both sides of the equation of the problem statement by \(4(\lambda xy + 1)\) and rearranging terms gives
\[
4x^2 - 4n^2\lambda xy = 4n^2 - 4y^2.
\]
Completing the square on the left side gives
\[
(2x - n^2\lambda y)^2 = 4n^2 - 4y^2 + n^4\lambda^2 y^2
\]
or
\[
(2x - n^2\lambda y)^2 - (n^4\lambda^2 - 4)y^2 = 4n^2. 
\] (1)
If \(n = 1\) and \(\lambda = 2\), the second term on the left vanishes and we have the infinite set of solutions \(x = y \pm 1\). With any other values for \(n\) and \(\lambda\), the multiplier of \(y^2\) in (1) is not a perfect square, so we have a Pell equation. One solution is \(x = n, y = n^3\lambda\), as can be shown by substitution into (1). And it is well known that a Pell equation with at least one solution has an infinite number of solutions.

Also solved by H.L. ABBOTT, University of Alberta; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.


The proposer also observes that, for \(\lambda n^2 > 2\), any solution of the given equation yields a right triangle of sides
\[
4\lambda n^2, (2x - \lambda n^2 y)n - 4, (2x - \lambda n^2 y)n + 4.
\]

\[\star\star\star\star\star\star\star\star\]


Let \(n\) be a positive integer and let \(\mathcal{P}_n\) be the set of ordered pairs \((a, b)\) of integers such that \(1 \leq a \leq b \leq n\). If \(f : \{1, 2, \ldots, n\} \to \mathbb{R}\) is an increasing function, and \(g : \mathcal{P}_n \to \mathbb{R}\) defined by
\[
g(a, b) = f(a) + f(b)
\]
is one-to-one, then \(g\) defines a (strict) total ordering \(\prec\) on \(\mathcal{P}_n\) by
\[
(a, b) \prec (c, d) \text{ if and only if } g(a, b) < g(c, d).
\]
Moreover \(\prec\) will have the property
\[
(a, b) \prec (c, d) \text{ whenever } a \leq c \text{ and } b \leq d \text{ (and } (a, b) \neq (c, d)) \text{ . (*)}
\]
Does every strict total ordering \(\prec\) of \(\mathcal{P}_n\) which satisfies (*) arise in this way?
Solution by Jean-Marie Monier, Lyon, France.

The answer is no, for $n = 4$ for example. Consider the strict total ordering $\prec$ defined on $\mathcal{P}_4$ by:

$$(1, 1) \prec (1, 2) \prec (2, 2) \prec (1, 3) \prec (1, 4) \prec (2, 3) \prec (2, 4) \prec (3, 3) \prec (2, 4) \prec (3, 4) \prec (4, 4).$$

$\prec$ satisfies $(\ast)$. Suppose there exists $f : \{1, 2, 3, 4\} \to \mathbb{R}$ and $g : \mathcal{P}_4 \to \mathbb{R}$ as above such that

$$(a, b) \prec (c, d) \iff g(a, b) < g(c, d).$$

Then we have

$$f(2) + f(2) = g(2, 2) < g(1, 3) = f(1) + f(3),$$

$$f(1) + f(4) < f(2) + f(3),$$

$$f(3) + f(3) < f(2) + f(4).$$

By summing we get a contradiction.

A similar counterexample was found by MARCIN E. KUCZMA, Warszawa, Poland; and by the proposer and the editor (jointly).

The problem was inspired by W.R. Ransom’s problem 3471 of the Amer. Math. Monthly, solution in Vol. 38 (1931) 474–475, which contains an example of an ordering (for $n = 4$) which does arise in the above way.

* * * * *


Let $P$ be an interior point of a triangle $ABC$ and let $AP, BP, CP$ intersect the circumcircle of $\triangle ABC$ again in $A', B', C'$, respectively. Prove that the power $p$ of $P$ with respect to the circumcircle satisfies

$$|p| \geq 4rr',$$

where $r, r'$ are the inradii of triangles $ABC$ and $A'B'C'$.

Solution by Murray S. Klamkin, University of Alberta.

We change the notation by letting $(A_1, A_2, A_3) = (A, B, C)$ and $(A'_1, A'_2, A'_3) = (A', B', C')$. For an interior point $P$, $p = R^2 - (OP)^2 \geq 0$, where $R$ is the circumradius and $O$ the circumcenter. Since $R_iR'_i = p$ for $i = 1, 2, 3$, where as usual $A_iP = R_i$ and $A'_iP = R'_i$, we have [from the similar triangles $A'_3A'_2P$ and $A_2A_3P$] that

$$\frac{a'_i}{a_i} = \frac{R'_2}{R_3} = \frac{R_1R_2R'_2}{R_1R_2R_3} = \frac{R_1p}{R_1R_2R_3},$$

where $a_i, a'_i$ are the sides of triangles $A_1A_2A_3$ and $A'_1A'_2A'_3$ respectively, and hence

$$a'_i = \frac{a_iR_iP}{K}$$

where $K = R_1R_2R_3$. So aside from the proportionality factor $p/K$, $\triangle A'_1A'_2A'_3$ is gotten by inversion from $\triangle A_1A_2A_3$ (see p. 294 of Mitrinović et al, *Recent Advances in Geometric*
Inequalities, or other references given re Crux 1514 on [1991: 117]). From the same references we see that, with \( F, F' \) the areas of triangles \( A_1A_2A_3 \) and \( A'_1A'_2A'_3 \) and \( r_i, r'_i \) the distances from \( P \) to their sides, and disregarding the factor \( p/K \),

\[
F' = \frac{1}{2} \sum a'_i r'_i = \frac{1}{2} \sum a_i r_i R_i^2 \quad \text{and} \quad r' = \frac{2F'}{\sum a_i R_i}.
\]

so that, accounting for the factor \( p/K \),

\[
r' = \frac{p}{K} \left( \frac{\sum a_i r_i R_i^2}{\sum a_i R_i} \right).
\]

It is also known that

\[
\sum a_i r_i R_i^2 = 2p F
\]

[equation (6) in M.S. Klamkin, An identity for simplexes and related inequalities, Simon Stevin 48 (1974–75) 57–64]. Hence the given inequality can be rewritten as

\[
P \geq \frac{4rp}{K} \left( \frac{\sum a_i r_i R_i^2}{\sum a_i R_i} \right)
\]

or

\[
R_1 R_2 R_3 \sum a_i R_i \geq 4r \sum a_i r_i R_i^2 = 8rp F.
\]

The latter now follows immediately from the product of the two known inequalities

\[
R_1 R_2 R_3 \geq 2rp
\]

(Crux 1327 [1989: 123]) and the Steensholt inequality

\[
\sum a_i R_i \geq 4F
\]

(item 12.19 of Bottema et al, Geometric Inequalities).

A related inequality is

\[
sR_1 R_2 R_3 \geq \sum a_i r_i R_i^2
\]


Also solved by the proposer.

* * * * *


Find a necessary and sufficient condition on reals \( c \) and \( d \) for the roots of \( x^3 + 3x^2 + cx + d = 0 \) to be in arithmetic progression.
Solution by Beatriz Margolis, Paris, France.

We claim that the necessary and sufficient condition is

\[ c - d = 2. \tag{1} \]

Assume the roots of the given equation to be \( a - r, a, a + r \). Relations between roots and coefficients yield

\[ (a - r)a(a + r) = -d, \tag{2} \]
\[ (a - r)a + a(a + r) + (a + r)(a - r) = c, \tag{3} \]
\[ (a - r) + a + (a + r) = -3. \tag{4} \]

Hence by (4) \( a = -1 \), so that by (2) and (3)

\[ 1 - r^2 = d \quad \text{and} \quad 3 - r^2 = c. \]

Therefore condition (1) is necessary.

Assume (1) holds. Then the given equation reads

\[ x^3 + 3x^2 + (2 + d)x + d = 0. \]

We see by inspection that \( x = -1 \) is a solution, so that we may factorize to obtain

\[ (x + 1)(x^2 + 2x + d) = 0. \]

Now the solutions to \( x^2 + 2x + d = 0 \) are \( x = -1 \pm \sqrt{1 - d} \). In other words, if (2) holds, the roots of the given equation are

\[ -1 - \sqrt{1 - d}, -1, -1 + \sqrt{1 - d}, \]

i.e., they form an arithmetic progression, and condition (2) is sufficient.

Observe that the necessary and sufficient conditions to have a real arithmetic progression are \( c - d = 2 \) and \( d \leq 1 \) (or \( c \leq 3 \)).
Lau points out that a more general result (replace the 3 by an arbitrary coefficient) appears as problem 7, section 1.5 of E.J. Barbeau, Polynomials, Springer-Verlag, 1989, with solution on p. 251.

\* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* 


The sequence $a_2, a_3, a_4, \ldots$ of real numbers is such that, for each $n$, $a_n > 1$ and the equation $[a_n x] = x$ has just $n$ different solutions. ($[x]$ denotes the greatest integer \leq x.) Find $\lim_{n \to \infty} a_n$.

Solution by Margherita Barile, student, Università degli Studi di Genova, Italy.

The only solution of $[a_1 x] = x$ must be $x = 0$. Let now $n > 1$. If $[a_n x] = x$ then $x \geq 0$, because $x < 0$ implies $a_n x < x$, since $a_n > 1$. As $[a_n 0] = 0$, $[a_n x] = x$ has exactly $n$ different solutions if and only if it has $n - 1$ different positive solutions.

Let $(a_n) = a_n - [a_n]$ (so $(a_n) \geq 0$). Then $a_n x = \left[a_n \right] x + (a_n) x$, thus for $x > 0$ an integer,

$$[a_n x] = x \iff x \leq [a_n] x + (a_n) x < x + 1$$

$$\iff [a_n] = 1 \text{ and } (a_n) x < 1,$$

which is true for exactly $n - 1$ different integers $x > 0$ if and only if

$$[a_n] = 1 \quad \text{and} \quad \frac{1}{n} \leq (a_n) < \frac{1}{n - 1}.$$

This implies

$$1 + \frac{1}{n} \leq a_n < 1 + \frac{1}{n - 1},$$

so that we immediately conclude

$$\lim_{n \to \infty} a_n = 1.$$

Note. The value of $a_1$ does not influence the result, but we can observe that the hypothesis is true for $a_1$ if $a_1 \geq 2$.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; RICHARD I. HESS, Rancho Palos Verdes, California; WALther JANOUS, Ursulengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incomplete solution was sent in.

Janous remarks that the same conclusion holds whenever the number of solutions of $[a_n x] = x$ goes to infinity with $n$.

\* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \* 


Determine an infinite class of integer triples $(x, y, z)$ satisfying the Diophantine equation

$$x^2 + y^2 + z^2 = 2yz + 2zx + 2xy - 3.$$
Solution by Hayo Ahlung, Benidorm, Spain.
The well-publicized identity (see Léo Sauvé’s footnote on [1976: 176])

\[
1^2 + (n^2 - n + 1)^2 + (n^2 + n + 1)^2 \\
= 2(n^2 - n + 1) + 2(n^2 - n + 1)(n^2 + n + 1) + 2(n^2 + n + 1) - 3 \tag{1}
\]

is one answer to this problem if we choose \( n \) to be any integer.

To find all solutions, we rearrange the original equation and get

\[
z^2 - 2(x + y)z + (x - y)^2 + 3 = 0
\]

and

\[
z = x + y \pm \sqrt{4xy - 3}.
\]

To make \( z \) an integer, \( 4xy - 3 \) must be the square of an odd number \( 2n+1 \), i.e., \( 4n^2 + 4n + 1 = 4xy - 3 \) or

\[
n^2 + n + 1 = xy,
\]

where \( n \) can be any integer. We can choose \( x \) and \( y \) as factors of \( n^2 + n + 1 \) (this can sometimes be done in several ways), and with

\[
z = x + y \pm (2n + 1)
\]

this solves the original equation. Due to the symmetry of the problem, any permutation of the values for \( x, y \) and \( z \) is also a solution. \( n, x, y \) and \( z \) can of course also have negative values. Using both signs for \( x, y, z \), and for \( 2n + 1 \) in the expression for \( z \) leads to duplications. But using + signs throughout already gives an infinite number of solutions. Factoring \( n^2 + n + 1 \) into \( x = 1 \) and \( y = n^2 + n + 1 \), and with \( z = x + y - (2n + 1) \), we get equation (1).

A nice special group is the series

\[
1, 1, 3, 7, 19, 49, 129, 337, 883, 2311, 6051, \ldots,
\]

where each term has the form \( F_{2n+1} - F_n F_{n+1} \) made up of Fibonacci numbers. Any three consecutive numbers of this series form a solution.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; ILIA BLASKOV, Technical University, Gabrovo, Bulgaria; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; GUO-GANG GAO, Université de Montréal; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; O. JOHNSON, student, King Edward’s School, Birmingham, England; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; JOHN H. LINDSEY, Northern Illinois University, Dekalb; STEWART METCHETTE, Culver City, California; JEAN-MARIE MONIER, Lyon, France; VEDULA N. MURTY, Penn State at
According to the proposer, the problem is equivalent to finding all triangles of sides \( \sqrt{x}, \sqrt{y}, \sqrt{z} \) \((x, y, z \text{ positive integers})\) with the same area as an equilateral triangle of unit sides.

\[
\star \quad \star \quad \star \quad \star \quad \star
\]

\[1562. \quad [1990; \quad 204] \quad \text{Proposed by Toshio Seimiya, Kawasaki, Japan.}
\]

Let \( M \) be the midpoint of \( BC \) of a triangle \( ABC \) such that \( \angle B = 2 \angle C \), and let \( D \) be the intersection of the internal bisector of angle \( C \) with \( AM \). Prove that \( \angle MDC \leq 45^\circ \).

I. Solution by Jordi Dou, Barcelona, Spain.

Consider the isosceles triangle \( A'BC \), where \( A' \) is on \( BA \) extended such that \( \angle BCA' = \angle CBA \). The centre \( O \) of the incircle \( \omega \) of \( A'BC \) is on \( CA \). Let \( \{E, F\} = \omega \cap CA \) (in the order \( CEOFA \) ), \( \{G, H\} = \omega \cap CD \) (in the order \( CGDH \) ), \( K = MF \cap CD \), \( L = ME \cap CD \), as shown.

![Diagram of the problem](image)

We have

\[
\angle FMH - \angle EMG = \angle FEH - \angle EHG = \angle ECH = \angle MCH = \angle MGH - \angle MHL ,
\]

therefore

\[
\angle MKG = \angle FMH + \angle MHL = \angle MGH + \angle EMG = \angle MLH.
\]

Since \( \angle FME = 90^\circ \), we conclude \( \angle MKL = \angle MLK = 45^\circ \). Thus

\[
\angle MDC = \angle MKL - \angle DMK \leq 45^\circ .
\]

Equality holds only when \( A = F \), i.e. \( A'BC \) is equilateral, i.e. \( \angle B = 60^\circ \), \( \angle C = 30^\circ \), \( \angle A = 90^\circ \).

Note: the solution is based on the nice property illustrated at the right, namely, if \( CD \) is the bisector of \( \angle C \), then \( KLM \) is an isosceles right triangle.
II. Solution by C. Festracks-Hamoir, Brussels, Belgium.

\[ \angle MDC = \angle MAC + \frac{C}{2}, \] ainsi

\[ \angle MDC \leq 45^\circ \iff \angle MAC \leq \frac{90^\circ - C}{2}. \]

Désignons par \( H \) le pied de la hauteur issue de \( A \), \( \angle HAC = 90^\circ - C \). Il faut donc démontrer que \( \angle MAC \leq \angle HAC/2 \), autrement dit que la bissectrice intérieure de \( \angle HAC \) coupe \( BC \) en un point \( K \) tel que \( KC \geq MC \). On a

\[ \frac{HK}{KC} = \frac{AH}{AC} = \sin C, \]

donc

\[ \frac{HC}{KC} = \frac{HK + KC}{KC} = \sin C + 1 \]
et

\[ KC = \frac{HC}{\sin C + 1} = \frac{b \cos C}{\sin C + 1}. \]

Ainsi

\[ KC \geq MC \iff \frac{b \cos C}{\sin C + 1} \geq \frac{a}{2} = \frac{b \sin A}{2 \sin B} = \frac{b \sin (B + C)}{2 \sin 2C} = \frac{b \sin 3C}{2 \sin 2C}, \]

\[ \iff 2 \cos C \sin 2C \geq (\sin C + 1) \sin 3C \]
\[ \iff \sin 3C + \sin C \geq \sin C \sin 3C + \sin 3C \]
\[ \iff 1 \geq \sin 3C, \]

ce qui est vrai.

III. Solution by the proposer.

Note first that

\[ 2 \angle MDC = 2(\angle MAC + \angle ACD) \]
\[ = 2 \angle MAC + \angle ACM \]
\[ = \angle MAC + \angle ABM. \] (1)

Let \( E \) be the intersection of the bisector of \( \angle B \) with \( AC \); then we get

\[ \angle EBC = \frac{1}{2} \angle ABC = \angle EBC. \]

As \( M \) is the midpoint of \( BC \), we get \( EM \perp BC \). Let \( F \) be the foot of the perpendicular from \( E \) to \( AB \). Since \( \angle ABE = \angle EBM \), we have \( EF = EM \). As \( EF \perp AB \) we have \( AE \geq EF \), therefore we get \( AE \geq EM \). Thus we obtain \( \angle AME \geq \angle MAE \), i.e., \( 90^\circ - \angle AMB \geq \angle MAC \). Therefore

\[ 90^\circ \geq \angle AMB + \angle MAC. \] (2)

From (1) and (2) we get \( 90^\circ \geq 2 \angle MDC \), consequently we have \( \angle MDC \leq 45^\circ \).
Also solved by HAYO AHLBURG, Benidorm, Spain; ILIA BLASKOV, Technical University, Gabrovo, Bulgaria; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; LJUBOMIR LJUBENOV, Stara Zagora, Bulgaria; VEDULA N. MURTY, Penn State Harrisburg; P. PENNING, Delft, The Netherlands; SHAILESH A. SHIRALI, Rishi Valley School, India; and D.J. SMEENK, Zaltbommel, The Netherlands.

* * * * * * *


Let \(N \geq 2\). For each positive integer \(n\) the number \(A_n > 1\) is implicitly defined by

\[
1 = \sum_{k=1}^{n} \frac{1}{A_n k^N - 1}.
\]

Show that the sequence \(A_1, A_2, A_3, \ldots\) converges.

Solution by John H. Lindsey, Northern Illinois University, Dekalb.

\(A_n > 1\) is defined since the given sum is decreasing in \(A_n\) and has limits \(\infty\) and 0 as \(A_n \to 1^+\) and \(A_n \to \infty\) respectively.

Suppose for some \(n\), \(A_{n+1} \leq A_n\). Then

\[
1 = \sum_{k=1}^{n} \frac{1}{a_n k^N - 1} \leq \sum_{k=1}^{n} \frac{1}{A_{n+1} k^N - 1} < \sum_{k=1}^{n+1} \frac{1}{A_{n+1} k^N - 1} = 1,
\]

a contradiction. Thus \(\{A_n\}\) increases, and it suffices to show \(A_n \leq 3\) for all \(n\). So suppose some \(A_n > 3\). Then

\[
1 = \sum_{k=1}^{n} \frac{1}{A_n k^N - 1} < \sum_{k=1}^{n} \frac{1}{3k^N - 1} \leq \sum_{k=1}^{n} \frac{1}{2k^N} \leq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2}
\]

\[
< \frac{1}{2} \left( 1 + \sum_{k=2}^{n} \frac{1}{(k-1)k} \right) = \frac{1}{2} \left( 1 + \sum_{k=2}^{n} \left( \frac{1}{k-1} - \frac{1}{k} \right) \right) = \frac{1}{2} \left( 1 + \frac{1}{n} \right) < 1,
\]

a contradiction.

Also solved by MARGHERITA BARILE, student, Università degli Studi di Genova, Italy; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; ROBERT B. ISRAEL, University of British Columbia; RICHARD KATZ, California State University, Los Angeles; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; and CHRIS WILDHAGEN, Rotterdam, The Netherlands.

Several solvers observed that the result is true for any \(N > 1\) and/or found better bounds for \(A_n\). Falkowitz points out that problem E3348 of the American Mathematical Monthly (1989, p. 735) can be shown to be similar to this problem.
From the set of vertices of the $n$-dimensional cube choose three at random. Let $p_n$ be the probability that they span a right-angled triangle. Find the asymptotic behavior of $p_n$ as $n \to \infty$.

**Solution by Shailesh A. Shirali, Rishi Valley School, India.**

Coordinatise $n$-space and assume without loss that the vertices of the $n$-dimensional cube are the $2^n$ possible points all of whose coordinates are 0 or 1. Let us now count the total number of 3-sets of vertices that can serve as the vertices of a right-angled triangle. This equals $2^n q(n)$ where $q(n)$ is the number of such 3-sets in which the “elbow” of the right angle is the origin $O(0, \ldots, 0)$. Now if $A, B$ are vertices of the unit cube and $\angle AOB = 90^\circ$, then the scalar product $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$. Considering the placement of 1’s in the two vectors (the 1’s must occur in disjoint positions if the scalar product is to be 0), it is clear that $q(n)$ can be equivalently defined as the total number of (unordered) pairs of nonempty disjoint subsets of a given $n$-set. It follows that

$$q(n) = \frac{1}{2} \sum_{r=1}^{n-1} C(n, r)(2^n-r-1),$$

where $C(n, r) = \frac{n!}{r!(n-r)!}$.

This summation is easily evaluated, for $\sum_{r=1}^{n-1} C(n, r) = 2^n - 2$ while by the binomial theorem

$$\sum_{r=1}^{n-1} C(n, r)2^{n-r} = \sum_{r=1}^{n-1} C(n, r)2^n = 3^n - 2^n - 1.$$

Therefore we find that

$$q(n) = \frac{1}{2}(3^n - 2 \cdot 2^n + 1).$$

Finally the required probability must equal

$$p(n) = \frac{2^n q(n)}{C(2^n, 3)} = \frac{3(3^n - 2 \cdot 2^n + 1)}{(2^n - 1)(2^n - 2)}.$$

Thus for large $n$, $p(n)$ is approximately equal to $3(3/4)^n$. In particular, $p(n)$ tends to 0 as $n$ tends to $\infty$.

Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALThER JANOUS, Ursulinenlymnasium, Innsbruck, Austria; JOHN H. LINDSEY, Northern Illinois University, Dekalb; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.
Find all Heronian triangles \( \triangle ABC \) (i.e., with integer sides and area) such that the lengths \( OA, AH, OH \) are in arithmetic progression, where \( O \) and \( H \) are respectively the circumcenter and the orthocenter of \( \triangle ABC \).

Solution by Vedula N. Murty, Penn State Harrisburg.

I claim that (with integer sides and area) a triangle satisfying the condition on \( OA, AH, OH \) cannot exist. Here is my proof.

We know

\[
OA = R \quad \text{(circumradius)}, \quad AH = 2R \cos A, \quad OH = R \sqrt{1 - 8 \cos A \cos B \cos C}
\]

(e.g., see Hobson, *A Treatise on Trigonometry*, Cambridge Univ. Press). If \( OA, AH, OH \) are to be in arithmetic progression, then we must have \( 2AH = OA + OH \), or

\[
4 \cos A = 1 + \sqrt{1 - 8 \cos A \cos B \cos C},
\]

and so

\[
(4 \cos A - 1)^2 = 1 - 8 \cos A \cos B \cos C,
\]

Therefore either \( \cos A = 0 \) or \( 1 - 2 \cos A = \cos B \cos C \).

If \( \cos A = 0 \) then \( A = \pi/2 \), so \( A \) and \( H \) coincide. In this case the lengths \( OA, AH, OH \) would be \( R, 0, R \) which are not in arithmetic progression.

Hence we must have \( 1 - 2 \cos A = \cos B \cos C \). Replacing \( \cos A, \cos B, \cos C \) by

\[
\frac{b^2 + c^2 - a^2}{2bc}, \quad \frac{c^2 + a^2 - b^2}{2ca}, \quad \frac{a^2 + b^2 - c^2}{2ab}
\]

respectively (where \( a, b, c \) are the sides of the triangle), and multiplying both sides by \( 4a^2bc \), we obtain

\[
4a^2bc - 4a^2(b^2 + c^2 - a^2) = (a^2 - (b^2 - c^2))(a^2 + (b^2 - c^2)) = a^4 - (b^2 - c^2)^2
\]

or

\[
3a^4 - 4a^2(b^2 + c^2 - bc) + (b^2 - c^2)^2 = 0.
\]

This quadratic equation in \( a^2 \) has two roots \( a^2 = (b - c)^2, \) \( a^2 = (b + c)^2/3, \) i.e. either \( a = b - c \) or \( c - b \), which is impossible as the triangle will be degenerate, or \( b + c = \sqrt{3}a \), which is also impossible if \( a, b, c \) are integers. Thus the problem as given has no solution!

Also solved by P. Penning, Delft, The Netherlands; and the proposer.

The above proof does not use that the area of the triangle is an integer. Penning (whose solution was the same) noticed this.

Hayo Ahlborg, Benidorm, Spain, looked at all permutations of the lengths \( OA, AH, OH \), but was not able to determine if any of them yield an arithmetic progression (in a Heronian triangle). Can the readers help?

* * * * *
Crux
Mathematicorum

Volume 17, Number 9  November 1991

CONTENTS

The Olympiad Corner: No. 129 ............................... R.E. Woodrow 257

Book Review .................................................... Andy Liu 268

Problems: 1681–1690 .............................................. 269

Solutions: 1564, 1567–1578 .................................. 271
THE OLYMPIAD CORNER

No. 129

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first item today is the 32nd I.M.O. held in Sigtuna, Sweden, July 17–18, 1991. My sources this year are Bruce Shawyer, Memorial University of Newfoundland, who was an observer for Canada and did not actually take part in the deliberations; a press release of the contest committee of the MAA; and Andy Liu, The University of Alberta, who did not attend the meeting, but who has excellent contacts. The results from him are relayed from Professor Pak-Hong Cheung, leader of the Hong Kong team. It would have been nice to include some more local colour from official representatives of the Canadian team, but their email messages to me went off into the ether.

This year a record 312 students from 54 countries were officially recorded as participants in the contest. The team from North Korea was disqualified because of overly striking similarities between the official solution of one problem and five of the submitted solutions. There were seven countries that did not send a full team of six persons. They were Denmark (5), Luxembourg (2), Switzerland (1), Trinidad & Tobago (4), Tunisia (4), Cyprus (4) and The Philippines (4).

The six problems of the competition were assigned equal weights of seven points each (the same as in the last 10 I.M.O.’s) for a maximum possible individual score of 42 (and a maximum possible team score of 252). For comparisons see the last 10 I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193], [1989: 193], and [1990: 193].

This year first place (gold) medals were awarded to the twenty students who scored 39 or higher. Second place (silver) medals were awarded to the 51 students whose scores were in the range 31–38, and third place (bronze) medals were awarded to the 84 students whose scores were in the range 19 to 30. In addition, as in recent years, Honourable Mention was given to any candidate who did not qualify for a medal, but who scored 7 out of 7 on one or more problems. There were nine perfect scores, four of which were by members of the winning Soviet team. One of these was by Evgeniya Malinnikova. This makes her fourth gold medal and her second perfect score at an I.M.O.!

Congratulations to the gold medal winners:

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<td>Perlin, Alexander</td>
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</table>
Here are the problems from this year’s I.M.O. Competition. Solutions to these problems along with those of the 1991 USA Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads 1991* which may be obtained for a small charge from:

Dr. W.E. Mientka  
Executive Director  
MAA Committee on H.S. Contests  
917 Oldfather Hall  
University of Nebraska  
Lincoln, Nebraska, USA 68588.

### 32nd INTERNATIONAL MATHEMATICAL OLYMPIAD

Sigtuna, Sweden  
First Day — July 17, 1991 (4 ½ hours)

1. Given a triangle $ABC$, let $I$ be the centre of its inscribed circle. The internal bisectors of the angles $A$, $B$, $C$ meet the opposite sides in $A'$, $B'$, $C'$ respectively. Prove that

$$
\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.
$$

2. Let $n > 6$ be an integer and $a_1, a_2, \ldots, a_k$ be all the natural numbers less than $n$ and relatively prime to $n$. If

$$
a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,
$$

prove that $n$ must be either a prime number or a power of 2.

3. Let $S = \{1, 2, \ldots, 280\}$. Find the smallest integer $n$ such that each $n$-element subset of $S$ contains five numbers which are pairwise relatively prime.
Second Day — July 18, 1991 (4 1/2 hours)

4. Suppose $G$ is a connected graph with $k$ edges. Prove that it is possible to label the edges $1, 2, 3, \ldots, k$ in such a way that at each vertex which belongs to two or more edges the greatest common divisor of the integers labelling those edges is equal to 1.

[A graph $G$ consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices $u, v$ belongs to at most one edge. The graph $G$ is connected if for each pair of distinct vertices $x, y$ there is some sequence of vertices $x = v_0, v_1, v_2, \ldots, v_m = y$ such that each pair $v_i, v_{i+1}$ $(0 \leq i < m)$ is joined by an edge of $G$.]

5. Let $ABC$ be a triangle and $P$ an interior point in $ABC$. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30^\circ$.

6. An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be bounded if there is a constant $C$ such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j| \cdot |i - j|^a \geq 1$$

for every pair of distinct non-negative integers $i, j$.

As the I.M.O. is officially an individual event, the compilation of team scores is unofficial, if inevitable. Team scores are obtained by adding up the individual scores of the members. These totals, as well as a breakdown of the medals awarded by country, is given in the following table.

Congratulations to the USSR for winning this year.

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<th>Prizes</th>
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</table>

This year the Canadian team slipped back from eleventh to fourteenth place, but put in a good performance. The team members, scores, and the leader of the Canadian team were:

Ian Goldberg 39 Gold
J.P. Grossman 36 Silver
Adam Logan 32 Silver
Peter Milley 22 Bronze
Mark von Raamsdonk 21 Bronze
Ka-Ping Yee 14 Honourable Mention.

Team leader: Professor Georg Gunther, Sir Wilfred Grenfell College.
The USA team slipped from 3rd to 5th. The results for its members were

Joel Rosenberg Gold
Kiran Kedlaya Silver
Robert Kleinberg Silver
Lenhard Ng Silver
Michael Sunitsky Silver
Ruby Breydo Bronze

The team leaders were Professors C. Rousseau, Memphis State University, and Dan Ullman of George Washington University.

* * *

Next we turn to “Archive” problems from the 1988 numbers of Crux.

Suppose \( S(X) \) is given by

\[
S(X) = X(1 + X^2(1 + X^3(1 + X^4(1 + \ldots )))).
\]

Is \( S(1/10) \) rational?

Solution by Murray S. Klamkin, University of Alberta.
We show that \( S(1/n) \) is irrational for any positive integer \( n > 1 \) where \( S(X) \) is the power series

\[
X + X^3 + X^6 + \cdots = \sum X^{n(n+1)/2}.
\]

Any rational number when expanded into a “decimal” in any base must eventually be periodic. Here \( S(1/n) \) in base \( n \) is .1010010001\ldots. By considering the blocks of successive zero digits, it is clear the number cannot be periodic.

Find the sum of the infinite series

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots
\]

where the terms are reciprocals of integers divisible only by the primes 2 or 3.

Correction and generalization by Murray S. Klamkin, University of Alberta.
One must add “except for the first term 1” at the end of the statement.

If we replace reciprocals of the numbers of the form \( 2^k3^l \) with reciprocals of numbers of the form \( P_1^{k_1}P_2^{k_2} \cdots P_n^{k_n} \) where \( P_1, P_2, \ldots, P_n \) are distinct primes, then the sum is given by

\[
\left(\sum_{k=0}^{\infty} \frac{1}{P_1^k}\right) \left(\sum_{k=0}^{\infty} \frac{1}{P_2^k}\right) \cdots \left(\sum_{k=0}^{\infty} \frac{1}{P_n^k}\right) = \frac{P_1}{P_1-1} \cdot \frac{P_2}{P_2-1} \cdots \frac{P_n}{P_n-1}.
\]

For the given problem this gives \( \frac{2}{1} \cdot \frac{3}{2} = 3. \)

[Editor’s Note: The problem was also solved by John Morvay, Springfield, Missouri.]
Next is an interesting solution to one of the problems of the 29th I.M.O. in Australia. Normally we don’t publish solutions to these problems as the “official” solutions are readily available. This warrants an exception.

6. [1988: 197] 29th I.M.O.

Let \( a \) and \( b \) be positive integers such that \( ab + 1 \) divides \( a^2 + b^2 \). Show that

\[
\frac{a^2 + b^2}{ab + 1}
\]

is the square of an integer.

**Solution by Joseph Zaks, The University of Haifa, Israel.**

If \( (a_i) \) is a doubly-infinite sequence (i.e., \(-\infty < i < \infty\)) of reals, satisfying

1. \( a_{i+2} = na_{i+1} - a_i \) for all \( i \), and
2. \( \frac{a_i^2 + a_{i+1}^2}{a_i a_{i+1} + 1} = n \) for one value of \( i \),

then (2) is satisfied for all values of \( i \).

To see it, observe that

\[
\frac{a_{i+2}^2 + a_{i+1}^2}{a_{i+2} a_{i+1} + 1} = \frac{(na_{i+1} - a_i)^2 + a_{i+1}^2}{(na_{i+1} - a_i)a_{i+1} + 1} = \frac{(a_{i+1}^2 + a_i^2) + n(na_{i+1}^2 - 2a_{i+1}a_i)}{(a_{i+1}a_i + 1) + (na_{i+1}^2 - 2a_{i+1}a_i)} = n
\]

where the last equality uses the assumption (2). Observe that \( a_{i+2} = na_{i+1} - a_i \) is equivalent to \( a_i = na_{i+1} - a_{i+2} \), thus it follows that (2) holds for all \( i \).

To solve the problem, suppose \( a \) and \( b \) are integers, for which \( (a^2 + b^2)/(ab + 1) = n \) is an integer. If \( a = b \), then it follows easily that \( a = b = 1 \), and the assertion is elementary. Otherwise, say \( 1 < a < b \), and let \( a_1 = a \) and \( a_2 = b \), and define \( (a_i) \) by \( a_{i+2} = na_{i+1} - a_i \) and by \( a_i = na_{i+1} - a_{i+2} \). It follows easily (since \( n > 1 \)) that the doubly infinite sequence of integers \( a_i \) is monotonic increasing (in \( i \)), thus in going backwards with \( i \), \( a_i \) tends to \(-\infty \). Thus \( a_i \) becomes negative for suitable \( i \). However, since \( n \) is positive, there is no \( j \) for which \( a_j > 0 \) and \( a_{j-1} < 0 \), because of (2). Thus \( a_{j-1} = 0 \) for some \( j \), implying (use (2) with \( i = j - 1 \)) that \( n = a_j^2 \).

As a by-product, we can get all the triples \( (a, b, n) \neq (1, 1, 1) \) of integers satisfying \( (a^2 + b^2)/(ab + 1) = n \). Such \( a \) and \( b \) will be any two consecutive terms of the sequence \( (a_i) \) satisfying \( a_0 = 0 \), \( a_1 = m \) and defined by (1) where \( n = m^2 \). The characteristic equation of (1) is \( x^2 - m^2x + 1 = 0 \), which has the solutions

\[
x = \frac{m^2 \pm \sqrt{m^4 - 4}}{2},
\]

thus, for suitable \( \alpha \) and \( \beta \),

\[
a_i = \alpha \left( \frac{m^2 + \sqrt{m^4 - 4}}{2} \right)^i + \beta \left( \frac{m^2 - \sqrt{m^4 - 4}}{2} \right)^i
\]
for all \(i \geq 0\). From \(a_0 = 0\) and \(a_1 = m\) we get

\[
a_i = \frac{m}{\sqrt{m^4 - 4}} \left( \frac{m^2 + \sqrt{m^4 - 4}}{2} \right)^i - \frac{m}{\sqrt{m^4 - 4}} \left( \frac{m^2 - \sqrt{m^4 - 4}}{2} \right)^i
\]

and the general solution is \((a_i, a_{i+1}, m^2)\) for \(i = 0, 1, \ldots\)

* * *

Now we turn to problems from the January 1990 number of the Corner. First let me apologize for leaving Murray Klamkin’s name off the list [1991: 235] of people who solved problem 5 of the Singapore Mathematical Society Interschool Mathematical Competition. His solution was stuck to one for the Chinese contests discussed below.


What necessary and sufficient conditions must real numbers \(A, B, C\) satisfy in order that

\[A(x - y)(x - z) + B(y - z)(y - x) + C(z - x)(z - y)\]

is nonnegative for all real numbers \(x, y\) and \(z\)?

*Solution by Murray S. Klamkin, University of Alberta.*

Expanding out, we get

\[Ax^2 + By^2 + Cz^2 + yz(A - B - C) + zx(B - C - A) + xy(C - A - B) \geq 0.\]

As is known, the corresponding matrix

\[
\begin{bmatrix}
A & \alpha & \beta \\
\alpha & B & \gamma \\
\beta & \gamma & C
\end{bmatrix}
\]

where \(2\alpha = C - A - B, 2\beta = B - C - A, 2\gamma = A - B - C\), must be nonnegative definite. The necessary and sufficient conditions for this are that all the principal minors be nonnegative. Thus \(A, B, C\) must be \(\geq 0\). All the principal 2nd order minors are the same and equal

\[
\frac{1}{4}(2(BC + CA + AB) - A^2 - B^2 - C^2).
\]

Since this expression corresponds to 4 times the square of the area of a triangle of sides \(\sqrt{A}, \sqrt{B}, \sqrt{C}\), these three numbers must satisfy the triangle inequalities. The third order minor, which is the determinant of the matrix, is 0 (just add the 2nd and 3rd row to the top one). Summarizing, the necessary and sufficient conditions are that \(\sqrt{A}, \sqrt{B}, \sqrt{C}\) are possible sides of a triangle, possibly degenerate.

*Comment:* On replacing \(A, B, C\) by \(a^2, b^2, c^2\), respectively, where \(a, b, c\) are sides of a triangle, the given form can be rewritten as

\[(ax)^2 + (by)^2 + (cz)^2 - 2(by)(cz) \cos \alpha - 2(cz)(ax) \cos \beta - 2(ax)(by) \cos \gamma\]
where \( \alpha, \beta, \gamma \) are the angles of the triangle. Since \( x, y, z \) are arbitrary, so are \( ax, by, cz \) so these can be replaced by arbitrary \( u, v, w \) to give

\[
u^2 + v^2 + w^2 - 2uvw \cos \alpha - 2wu \cos \beta - 2uv \cos \gamma.
\]

For a generalization of this nonnegative form and an expression of it as a sum of two squares, see \textit{Crux} 1201 [1988: 90].

[Editor’s Note: Seung-Jin Bang, Seoul, Republic of Korea, also sent in a solution that gave \( A, B, C \geq 0 \) and \( A^2 + B^2 + C^2 \leq 2(AB + BC + CA) \) as necessary and sufficient conditions, but that did not extend to the observation about triangles. He sets \( X = x - y \) and \( Y = x - z \), and the expression becomes

\[
\]

Setting \( X = 0, Y = 0 \) and then \( X = Y \) gives \( A, B, C \geq 0 \). Viewing the expression as a quadratic in \( X \) which is non-negative gives \( (A - B - C)^2 - 4BC \leq 0 \) so that \( A^2 + B^2 + C^2 \leq 2(AB + BC + CA) \).


Determine all functions \( f \) from the rational numbers to the complex numbers such that

(i) \( f(x_1 + x_2 + \cdots + x_{1988}) = f(x_1)f(x_2) \cdots f(x_{1988}) \)

for all rational numbers \( x_1, x_2, \ldots, x_{1988} \), and

(ii) \( f(1988)f(x) = f(1988)f(x) \)

for all rational numbers \( x \), where \( \overline{z} \) denotes the complex conjugate of \( z \).


Suppose \( f \neq 0 \). Since \( f(x) = f(x)f(0)^{1987} \), we have \( (f(0))^{1987} = 1 \). Note that \( f(x + y) = f(x)f(y)f(0)^{1986} \). Let \( g(x) = f(x)/f(0) \). Then

\[
g(x + y) = \frac{f(x + y)}{f(0)} = \frac{f(x)f(y)f(0)^{1986}}{f(0)} = \frac{f(x)}{f(0)} \cdot \frac{f(y)}{f(0)} \cdot (f(0))^{1987} = g(x)g(y).
\]

It follows easily that \( g(x) = e^{bx} \) for some complex number \( b \). From this we get that \( f(x) = f(0)e^{bx} \). Since the condition (ii) implies \( e^{b - \overline{b}}(x - 1988) = 1 \) we obtain that \( b \) must be real. Answer: \( f \equiv 0 \) or \( f(x) = ae^{bx} \) where \( a^{1987} = 1 \) and \( b \) is real.


In triangle \( ABC \), angle \( C \) is \( 30^\circ \). \( D \) is a point on \( AC \) and \( E \) is a point on \( BC \) such that \( AD = BE = AB \). Prove that \( OI = DE \) and \( OI \) is perpendicular to \( DE \), where \( O \) and \( I \) are respectively the circumcentre and incentre of triangle \( ABC \).

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

We generalize the problem by considering an arbitrary triangle \( ABC \) with \( c < a \), \( c < b \). The projection of \( O \) on \( CB \) is \( F \), that on \( CA \) is \( G \). The projection of \( I \) on \( CB \) is \( H \), that on \( CA \) is \( K \). Now

\[
HF = BF - BH = \frac{1}{2}a - (s - b) = \frac{1}{2}(b - c) = \frac{1}{2}CD.
\]
and $KG = \frac{1}{2}(a - c) = \frac{1}{2}CE$. The production of $GO$ intersects $IH$ (or its production) in $L$. Consider triangle $OIL$. Angle $\angle ILO = \gamma (= \angle C)$, since $OL \perp CA$, $IL \perp CB$. It is easy to verify that

$$IL = \frac{KG}{\sin \gamma} = \frac{CE}{2 \sin \gamma}$$

and

$$OL = \frac{HF}{\sin \gamma} = \frac{CD}{2 \sin \gamma}.$$  

We see that triangles $LIO$ and $CED$ are similar, for $LI : LO = CE : CD$. Moreover $LI \perp CE$ and $LO \perp CD$ imply $IO \perp ED$. Also

$$IO = \frac{DE}{2 \sin \gamma},$$

and for $\gamma = 30^\circ$ this gives $IO = DE$, as desired.

* 

The last solution to a problem in the January number received from readers is the following.


Define $x_n = 3x_{n-1} + 2$ for all positive integers $n$. Prove that an integer value can be chosen for $x_0$ such that 1988 divides $x_{100}$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and by Murray S. Klamkin, University of Alberta.

Letting $x_n = 3^n y_n$, we obtain the geometric difference equation $y_n = y_{n-1} + 2/3^n$. Thus

$$y_n = y_0 + \frac{2}{3} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-1}} \right)$$

so that

$$x_n = (x_0 + 1)3^n - 1.$$  

We now want $x_{100} = (x_0 + 1)3^{100} - 1$ to be divisible by 1988. Now let $3^{100} = 1988k + r$ where $0 < r < 1988$. Since 1988 is not divisible by 3, $r$ is relatively prime to 1988. It now suffices to show the existence of integers $x_0$ and $\ell$ such that $r(x_0 + 1) - 1 = 1988\ell$. Since $(r, 1988) = 1$ there are an infinite number of desired integer pairs $(x_0, \ell)$. Indeed since $1988 = 4 \cdot 7 \cdot 71$, we must have $x_{100} \equiv 0 \mod 4$, $x_{100} \equiv 0 \mod 7$ and $x_{100} \equiv 0 \mod 71$; it follows that $x_0 \equiv 0 \mod 4$, $x_0 \equiv 1 \mod 7$, and $x_0 \equiv 45 \mod 71$. By the Chinese Remainder Theorem we have $x_0 \equiv 400 \mod 1988$. 
We now turn to problems from the February 1990 number of the Corner. Here are solutions to the first five problems from the XIV “ALL UNION” Mathematical Olympiad (U.S.S.R.) [1990: 33–34]. A frequently occurring team solver is JACL, an acronym for two Edmonton students, Jason A. Colwell of Old Scona School and Calvin Li of Archbishop MacDonald School, and Andy Liu of The University of Alberta.

1. All the two-digit numbers from 19 to 80 are written in a row. The result is read as a single integer 19202122...787980. Is this integer divisible by 1980?

   Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by JACL; by Stewart Metchette, Culver City, California; by John Morvay, Springfield, Missouri; by Bob Prielipp, University of Wisconsin–Oshkosh; by Don St. Jean, George Brown College, Toronto; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

   Since 1980 = 2^3 · 3 · 5 · 11, we only need check divisibility by 4, 9, 5 and 11. Since the last two digits are 8 and 0, the number is divisible by both 4 and 5. The sum of the digits in the odd positions is 1 + (2 + 3 + 4 + 5 + 6 + 7)10 + 8 = 279. The sum of those in the even positions is

   \[ 9 + (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)6 + 0 = 279. \]

   Since 279 + 279 = 558 is divisible by 9, so is the number. Since 279 − 279 = 0 is divisible by 11, so is the number. Hence the number is divisible by 1980.

2. Side \(AB\) of a square \(ABCD\) is divided into \(n\) segments in such a way that the sum of lengths of the even numbered segments equals the sum of lengths of the odd numbered segments. Lines parallel to \(AD\) are drawn through each point of division, and each of the \(n\) “strips” thus formed is divided by diagonal \(BD\) into a left region and a right region. Show that the sum of the areas of the left regions with odd numbers is equal to the sum of the areas of the right regions with even numbers.

   Solutions by JACL; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

   We show that the conclusion holds when \(ABCD\) is any rectangle. Let \(AD = a\) and \(AB = b\). Denote by \(h_i\) the height of the \(i\)th strip, so that \(\sum_{i \text{ odd}} h_i = \sum_{i \text{ even}} h_i\) and \(\sum_{i = 1}^{n} h_i = b\). Also let \(L_i\) and \(R_i\) denote the area of the \(i\)th left region and right region, respectively, \(i = 1, 2, \ldots, n\). Then

   \[
   \sum_{i \text{ odd}} L_i + \sum_{i \text{ even}} L_i = \frac{1}{2}ab
   \]

   and

   \[
   \sum_{i \text{ even}} L_i + \sum_{i \text{ even}} R_i = a \sum_{i \text{ even}} h_i = \frac{1}{2}ab.
   \]
From these we immediately get
\[ \sum_{i \text{ odd}} L_i = \sum_{i \text{ even}} R_i. \]

3. A payload, packed into containers, is to be delivered to the orbiting space station “Salyut”. There are at least 35 containers, and the total payload weighs exactly 18 tons. Seven “Progress” transport ships are available, each of which can deliver a 3-ton load. It is known that these ships altogether can (at least) carry any 35 of the containers at once. Show that in fact they can carry the entire load at once.

Solution by JACL.

Arrange the containers in non-increasing order of weight. Let \( n \) be the largest integer such that all the containers from the first to the \( n \)-th can be carried by the ships. By hypothesis \( n \geq 35 \). Suppose there is another container of weight \( \omega \) tons. Since it cannot be carried as well, each ship must already be carrying more than \( 3 - \omega \) tons. However, the total weight being carried so far is at most \( 18 - \omega \). Hence \( 7(3 - \omega) < 18 - \omega \) or \( \omega > 1/2 \). Each of the \( n \) containers being carried weighs at least \( \omega \) tons, and \( n \geq 35 \). Counting the extra container, the total weight exceeds \( 36\omega > 18 \) tons, which is a contradiction.

4. Points \( M \) and \( P \) are the midpoints of sides \( BC \) and \( CD \) of convex quadrilateral \( ABCD \). If \( AM + AP = a \), show that the area of the region \( ABCD \) is less than \( a^2/2 \).

Solution by JACL.

Denote by \([P]\) the area of polygon \( P \). Since \( BM = CM \), \([ABM] = [ACM]\). Similarly \([ADP] = [ACP] \) so that \([ABCD] = 2[AMP] \). Now \( MP \) is parallel to \( BD \), and \( C \) is at the same distance from \( MP \) as \( BD \) is from \( MP \). Since \( ABCD \) is convex, \( A \) is on the opposite side to \( C \) of \( BD \). Now \( AMP \) and \( CMP \) have the same base. Since \( AMP \) has the greater altitude, \([AMP] > [CMP]\). Finally, note that
\[ [AMP] = \frac{1}{2} AM \cdot AP \sin MAP \leq \frac{1}{2} AM \cdot AP \leq \frac{1}{8} a^2 \]
by the Arithmetic-Mean Geometric-Mean Inequality. Hence
\[ [ABCD] = 2[AMP] = 2[AMP] + 2[AMP] < 4[AMP] \leq \frac{1}{2} a^2 \]
as desired.

5. Does the equation \( x^2 + y^3 = z^4 \) have solutions for prime numbers \( x, y \) and \( z \)?

Solutions by Dieter Bennewitz, Koblenz, Germany; by JACL; by John Morway, Springfield, Missouri; by Bob Prielipp, University of Wisconsin-Oshkosh; by Don St. Jean, George Brown College, Toronto; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. (The solution given is the one sent in by Bob Prielipp.)

We shall show that the equation has no solutions for a positive integer \( x \) and prime numbers \( y \) and \( z \).

Suppose on the contrary that there is a solution of this form. The equation is equivalent to
\[ y^3 = (z^2 - x)(z^2 + x). \]
Because \(y\) is a prime number and \(x\) is a positive integer, \(z^2 - x = 1\) and \(z^2 + x = y^3\) or \(z^2 - x = y\) and \(z^2 + x = y^2\).

If \(z^2 - x = 1\) and \(z^2 + x = y^3\), then
\[
2z^2 = y^3 + 1 = (y + 1)(y^2 - y + 1).
\]

It follows that \(y \neq 2\), so that \(y\) is an odd prime number. Thus \(y > 2\), making \(y + 1 > 3\) and \(y^2 - y + 1 > y + 1\). Since \(z\) is a prime number we now obtain \(z = y + 1\) and \(y^2 - y + 1 = 2z\).

But \(z = y + 1\) where \(y\) and \(z\) are prime numbers with \(y > 2\) impossible.

If \(z^2 - x = y\) and \(z^2 + x = y^2\) then
\[
2z^2 = y^2 + y = y(y + 1).
\]

It follows that \(y \neq 2\), so \(y\) is an odd prime. Thus \(y > 2\). Since \(y < y + 1\) and \(z\) is a prime we get \(y = z\) and \(y + 1 = 2z\). This gives \(z = 1\), a contradiction.

Comment by Murray S. Klamkin, University of Alberta.

There are however an infinite number of solutions if we remove the prime number restrictions, and even for the more general equation
\[
x^r + y^s = z^t \quad \text{where} \quad (rs, t) = 1.
\]

Just let \(x = 2^{mr}\), \(y = 2^{mr}\) and \(z = 2^p\). Then we must satisfy \(mrs + 1 = pt\). Since \(rs\) is relatively prime to \(t\), there exist an infinite number of positive integer pairs \((m, p)\) satisfying the conditions.

* * * * *

That's all the space available for this issue! Send me your contests and solutions.

* * * * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.


This book is a collection of the 213 problems set in the first twelve years of the “Mathematical Challenges”, organized by the Scottish Mathematical Council. Unfortunately, the preface gives no description of the organization and philosophy of the “Mathematical Challenges”. The readers should bear in mind that this is not a conventional contest, where participants write the papers in time-limited sittings. Instead, four sets of problems are sent out during the school year, and the students have over a month to think about them.
Many of the problems have a strong recreational flavour, a welcome relief from the usual “solid and stolid” mode of British offerings. There is a great variety, including cryptarithms, cross-numbers, logical inferences and number patterns. There is also a healthy dosage of problems in geometry, as well as in diophantine equations, inequalities and other Olympiad topics. Some of them are familiar classics, but most are new to the reviewer.

Here is a sample problem. “Four players successively select two cards from four numbered cards face down on a table. The total values of the cards drawn are 6, 9, 12 and 15. Then two of the four cards are turned over and their total value is found to be 11. Determine the value of each of the other two cards.”

The problems are primarily at the pre-Olympiad level, even though a few prove rather difficult. The reviewer feels strongly that there is an urgent need for a book such as this. Many aspiring young students who may be scared off by the Olympiad level problems will find the ones in this book attractive. Once they have gained some proficiency and confidence, they will look for further challenges.

Although no names are associated with the book, it is no secret that the driving force behind the “Mathematical Challenges” is the indefatigable Dr. David Monk of the University of Edinburgh. To initiate such an enrichment program for the students is an admirable feat, but to have sustained the effort over such a long period displays conviction and dedication at the highest level.

For readers in North America, there is unfortunately no local distributor. All orders should be placed directly with Blackie and Son Limited at 7 Leicester Place, London, WC2H7BP, United Kingdom. The current price is £8.50 , plus £3.81 for air mail or £1.26 for surface mail.

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1992, although solutions received after that date will also be considered until the time when a solution is published.
1681. Proposed by Toshio Seimiya, Kawasaki, Japan.

$ABC$ is an isosceles triangle with $\overline{AB} = \overline{AC} < \overline{BC}$. Let $P$ be a point on side $BC$ such that $\overline{AP}^2 = \overline{BC} \cdot \overline{PC}$, and let $CD$ be a diameter of the circumcircle of $\triangle ABC$. Prove that $\overline{DA} = \overline{DP}$.

1682*. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

For a finite set $S$ of natural numbers let

$$Alt(S) = x_1 - x_2 + x_3 - \cdots,$$

where $x_1 > x_2 > x_3 > \cdots$ are the elements of $S$ in decreasing order. Determine

$$f(n) = \sum Alt(S),$$

where the sum is extended over all non-empty subsets $S$ of $\{1, 2, \ldots, n\}$.


Given is a fixed triangle $ABC$ and fixed positive angles $\mu, \nu$ such that $\mu + \nu < \pi$.

For a variable line $l$ through $C$, let $P$ and $Q$ be the feet of the perpendiculars from $A$ and $B$, respectively, to $l$, and let $Z$ be such that $\angle ZPQ = \mu$ and $\angle ZQP = \nu$ (and, say, the sense of $QPZ$ is clockwise). Determine the locus of $Z$.


Let

$$f(x, y, z) = x^4 + x^3z + ax^2z^2 + bx^2y + cxyz + y^2.$$

Prove that for any real numbers $b, c$ with $|b| > 2$, there is a real number $a$ such that $f$ can be written as the product of two polynomials of degree 2 with real coefficients; furthermore, if $b$ and $c$ are rational, $a$ will also be rational.

1685. Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.

If equilateral triangles $A_2A_3P_1, A_3A_1P_2, A_1A_2P_3$ are erected externally on the sides of a triangle $A_1A_2A_3$, then $A_1P_1, A_2P_2, A_3P_3$ concur at a point $R$ called the isogonic center (see p. 218 of R.A. Johnson, Advanced Euclidean Geometry). Prove that the line joining $R$ and its isogonal conjugate is parallel to the Euler line of the triangle.

1686. Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.

The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 4/3$ and

$$a_{n+1} = \frac{3(5 - 7a_n)}{2(10a_n + 17)}$$

for $n \geq 0$. Find a formula for $a_n$ in terms of $n$.


The octahedron $ABCDEF$ is such that the three space diagonals $AF, BD, CE$ meet at right angles. Show that


where $[XYZ]$ is the area of triangle $XYZ$. 
Solve the equation
\[ a \log b = \log(ab) \]
if \(a\) and \(b\) are required to be (i) positive integers, (ii) positive rational numbers.

1689. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

\(AA'\) is a diameter of circle \(C = (O, r)\). Two congruent circles \(C_1 = (O_1, a)\) and \(C_2 = (O_2, a)\) \((a < r)\) are internally tangent to \(C\) at \(A\) and \(A'\) respectively. In one half of the circle \(C\) we draw two more circles \((O_3, b)\) and \((O_4, c)\) externally touching each other, both internally touching \(C\), and also externally touching \(C_1\) and \(C_2\) respectively. Show that

(i) \(r = a + b + c\);  
(ii) \(O_3O_4 \parallel AA'\).

1690. *Proposed by Charlton Wang, student, Waterloo Collegiate Institute, and David Vaughan and Edward T.H. Wang, Wilfrid Laurier University.*

When working on a calculus problem, a student misinterprets “the average rate of change of \(f(x)\) from \(a\) to \(b\)” to mean “the average of the rates of change of \(f(x)\) at \(a\) and \(b\)”, but obtains the correct answer. Determine all infinitely differentiable functions \(f(x)\) for which this occurs, i.e., for which

\[
\frac{f(b) - f(a)}{b - a} = \frac{f'(b) + f'(a)}{2}
\]
for all \(a \neq b\).

**SOLUTIONS**

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

**1564. [1990: 205] Proposed by Jordi Dou, Barcelona, Spain.**

Given three pairs of points \((P, P'), (Q, Q'), (R, R')\), each pair isogonally conjugate with respect to a fixed unknown triangle, construct the isogonal conjugate \(X'\) of a given point \(X\).

Solution by the proposer.

The construction relies on classical theorems that can be found in the older projective geometry texts or in Pierre Samuel’s recent *Projective Geometry* (Springer, Undergraduate Texts in Mathematics, 1988), see especially sections 2.4 (pp. 64–65) and 4.2. The crux is that \(X\) and \(X'\) are isogonal conjugates with respect to \(\Delta ABC\) if and only if they are conjugate with respect to each conic of the pencil \(\Omega\) determined by the four points \(I, I_A, I_B, I_C\), the incentre and excentres of \(\Delta ABC\). These conics are equilateral hyperbolas. We know that the polars of a point \(X\) with respect to the conics of a pencil all pass through a point \(X'\); this is the conjugate of \(X\) that we are after. In order to resolve the problem it is sufficient to determine two hyperbolas \(\omega_1, \omega_2\) of \(\Omega\) and construct the polars \(x_1, x_2\) of \(X\) with respect to \(\omega_1, \omega_2\). Their intersection is then \(X'\).
We put $PP' = p$, $QQ' = q$, $RR' = r$, $p \cap q = N$, $q \cap r = L$, $r \cap p = M$. Let $P_N$ and $P_M$ be the harmonics of $N$ and $M$ with respect to $P, P'$, and analogously define $Q_L, Q_N$ and $R_M, R_L$. Let $Q_NR_M = p'$, $R_LP_N = q'$, $P_MQ_L = r'$, and put $p' \cap q' = N'$, $q' \cap r' = L'$, $r' \cap p' = M'$. The pairs $PP', QQ', RR'$ are then conjugates with respect to every one of the degenerate conics $[p, p']$, $[q, q']$, $[r, r']$. All the conics which pass through $N, P_N, N', Q_N$ (the intersections of $p, p'$ and $q, q'$) have as conjugate points the three pairs $PP', QQ', RR'$, and analogously for the conics which pass through $[p, p'] \cap [r, r'] = \{L, Q_L, L', R_L\}$ and through $[r, r'] \cap [q, q'] = \{M, R_M, M', P_M\}$. Let $H_1$ be the orthocentre of $NP_NQ_N$. The conic $\omega_1 = [NP_NNP_NQ_NH_1]$ is an equilateral hyperbola and has as conjugates the pairs $PP', QQ', RR'$, and analogously for the conic $\omega_2 = [LQ_LR_LH_2]$, $H_2$ being the orthocentre of $LQ_LR_L$.

Using Pascal’s theorem, construct the other intersections $Y_1$ and $Z_1$ of the lines $XN'$ and $XP_N$ with $\omega_1$. The polar $x_1$ of $X$ with respect to $\omega_1$ is the line joining the points $Y_1Z_1 \cap N'P_N$ and $Y_1P_N \cap Z_1N'$. Analogously construct the polar $x_2$ of $X$ with respect to $\omega_2$. Finally, $X' = x_1 \cap x_2$.

Since all constructions in the solution are with straightedge alone, there exists a unique solution in general. On the other hand, the determination of the quadrilateral $H_1H_1H_2H_2 = \omega_1 \cap \omega_2$ (and of the diagonal points $A, B, C$) involves finding the points of intersection of two conics — i.e., solving a fourth degree equation. They cannot generally be constructed with Euclidean tools.

I have not found a solution for the analogous problem where “isogonal” is replaced by “isotomic”.

* * * * * * *


Let

$$f(x_1, x_2, \ldots, x_n) = \frac{x_1 \sqrt{x_1} + \cdots + x_n}{(x_1 + \cdots + x_{n-1})^2 + x_n}.$$ 

Prove that $f(x_1, x_2, \ldots, x_n) \leq \sqrt{2}$ under the condition that $x_1 + \cdots + x_n \geq 2$ and all $x_i \geq 0$.

Solution by Walther Janous, Ursulengymnasium, Innsbruck, Austria.

Let $k \geq 1$. We consider more generally $f(x_1, \ldots, x_n)$ where $x_i \geq 0$ for all $i$ and $x_1 + \cdots + x_n \geq k$, and show

$$f(x_1, \ldots, x_n) \leq \frac{1}{2 - 1/\sqrt{k}}.$$ 

Put $a = x_1, b = x_2 + \cdots + x_{n-1}, c = x_n$. Then

$$f(x_1, \ldots, x_n) = \frac{a \sqrt{a} + b + c}{(a + b)^2 + c} = g(a, b, c),$$

where $a, b, c \geq 0, a + b + c \geq k$. We put $a + b + c = s, s \geq k, s$ fixed for the moment. Then

$$g(a, b, c) = \frac{a \sqrt{s}}{(s - c)^2 + c} \leq \frac{(s - c) \sqrt{s}}{(s - c)^2 + c} = \frac{\sqrt{s}}{(s - c) + c/(s - c)}.$$
Now
\[(s - c) + \frac{c}{s - c} \geq 2\sqrt{s} - 1,
\]
as this inequality is equivalent to \((c - s + \sqrt{s})^2 \geq 0\). Therefore
\[g(a, b, c) \leq \frac{\sqrt{s}}{2\sqrt{s} - 1} = \frac{1}{2 - 1/\sqrt{s}}.
\]
Since the term on the right hand side decreases, we get
\[g(a, b, c) \leq \frac{1}{2 - 1/\sqrt{k}},
\]
as was to be shown. There holds equality for \(b = 0, c = k - \sqrt{k}, a = \sqrt{k}\), i.e. for \(x_1 = \sqrt{k}, x_2 = \cdots = x_{n-1} = 0, x_n = k - \sqrt{k}\). For the original problem \((k = 2)\) we get the better bound
\[f(x_1, \ldots, x_n) \leq \frac{\sqrt{2}}{2\sqrt{2} - 1} = \frac{4 + \sqrt{2}}{7}.
\]

Also solved by MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; JOHN H. LINDSEY, Northern Illinois University, DeKalb; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Kuczma, Lau and Lindsey all obtained the best possible bound given above.

* * * * *

Show that
\[
\sum \sin A \geq \frac{2}{\sqrt{3}} \left( \sum \cos A \right)^2
\]
where the sums are cyclic over the angles \(A, B, C\) of an acute triangle.

I. Solution by Vedula N. Murty, Penn State Harrisburg.
The inequality proposed can be restated as
\[y \geq \frac{2}{\sqrt{3}} (1 + x)^2\]
where
\[y = \frac{s}{R} = \sum \sin A, \quad 1 + x = 1 + \frac{r}{R} = \sum \cos A,
\]
\(s, r, R\) being the semiperimeter, inradius and circumradius respectively. In the paper “A new inequality for \(R, r,\) and \(s\)”, published in Crux [1982: 62–68], I have presented a picture showing the Type I and Type II triangle regions. The curve
\[y = \frac{2}{\sqrt{3}} (1 + x)^2\] (1)
is a parabola opening upwards with vertex at the point \( M(-1, 0) \) shown in my picture. It cuts the \( y \)-axis at \((0, 2/\sqrt{3})\), which is below the point \( E(0, \sqrt{3}) \), and passes through \( A(1/2, 3\sqrt{3}/2) \). Thus the inequality clearly holds for all Type I triangles since the corresponding region (vertical shading) is above the curve (1). It is also clear that there exist obtuse Type II triangles for which the proposed inequality does not hold. However, from the classic Steinig inequality
\[
y^2 \geq 16x - 5x^2
\]
(item 5.8 in Bottema et al, Geometric Inequalities), we see that the proposed inequality holds for all \( x \) values for which
\[
16x - 5x^2 \geq \frac{4}{3}(1 + x)^4,
\]
i.e.,
\[
4x^4 + 16x^3 + 39x^2 - 32x + 4 \leq 0.
\]
The above polynomial has two real roots \( x_0 \approx 0.157 \) and \( x_1 = 0.5 \), and two complex conjugate roots. Hence for \( x_0 \leq x \leq 1/2 \) the proposed inequality holds. [Editor’s note. From Murty’s paper, all acute Type II triangles correspond to values \( x \geq 1/4 \), thus all acute Type II triangles satisfy the proposed inequality. Alternatively, since point \( F \) in Murty’s picture has \( y \)-coordinate \( \sqrt{71 - 8\sqrt{2}/4} \), one need only check that
\[
\frac{2}{\sqrt{3}}(1 + \frac{1}{4})^2 < \frac{4}{3}\sqrt{71 + 8\sqrt{2}},
\]
since then the curve (1) must pass below \( F \).]

II. Solution by the proposer.

From
\[
\sum \cos A = 1 + 4 \prod \sin(A/2)
\]
we have
\[
(\sum \cos A)^2 = 1 + 8 \prod \sin(A/2) + 16 \prod \sin^2(A/2)
\leq 1 + 8 \prod \sin(A/2) + 16(1/8) \prod \sin(A/2)
= 1 + 10 \prod \sin(A/2).
\]
So a sharper inequality to prove is
\[
\sum \sin A \geq \frac{2}{\sqrt{3}} \left(1 + 10 \prod \sin(A/2)\right).
\]
(2)

Now
\[
\sum \sin A = \frac{s}{R} \quad \text{and} \quad \prod \sin(A/2) = \frac{r}{4R}
\]
so from (2) we have to prove that \( \sqrt{3}s \geq 2R + 5r \), or
\[
3s^2 \geq 4R^2 + 20Rr + 25r^2.
\]
It is known (Crux 999 [1986: 80]) that for acute triangles,

\[ s^2 \geq 2R^2 + 8Rr + 3r^2, \]

so it suffices to show that

\[ 6R^2 + 24Rr + 9r^2 \geq 4R^2 + 20Rr + 25r^2, \]

which simplifies to

\[ (R + 4r)(R - 2r) = R^2 + 2Rr - 8r^2 \geq 0, \]

which is true.

Also solved by WALTHER JANOUS, Ursulinenygmnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; and JOHN H. LINDSEY, Northern Illinois University, DeKalb.

Kuczma also showed that the inequality holds for some obtuse triangles, namely for any triangle whose angles do not exceed \( \approx 132.40' \). Janous made the substitution \( A \rightarrow (\pi - A)/2 \), etc., in the given inequality to obtain

\[ \sum \cos \frac{A}{2} \geq \frac{2}{\sqrt{3}} \left( \sum \sin \frac{A}{2} \right)^2, \]

valid for all triangles.

* * * * *


Evaluate

\[ \int_0^a \frac{\ln(1 + ax)}{1 + x^2} \, dx \]

where \( a \) is a constant.

I. Solution by Margaret Izienicki, Mount Royal College, Calgary.

Let \( F(a) \) be the given integral. By differentiation with respect to \( a \) under the integral sign, we get

\[
\frac{d}{da} F(a) = \int_0^a \frac{x \, dx}{(1 + ax)(1 + x^2)} + \frac{\ln(1 + a^2)}{1 + a^2} \quad (F(0) = 0) \\
= \frac{1}{1 + a^2} \int_0^a \left( \frac{x}{1 + x^2} + \frac{a}{1 + x^2} - \frac{a}{1 + ax} \right) \, dx + \frac{\ln(1 + a^2)}{1 + a^2} \\
= \frac{1}{1 + a^2} \left( \frac{1}{2} \ln(1 + a^2) + a \tan^{-1} a \right) \\
= \frac{d}{da} \left( \frac{1}{2} \ln(1 + a^2) \cdot \tan^{-1} a \right).
\]
Integrating with respect to $a$, and applying $F(0) = 0$, we get

$$F(a) = \frac{1}{2} \ln(1 + a^2) \cdot \tan^{-1} a.$$ 

II. Solution by Maria Mercedes Sánchez Benito, I.B. Luis Bunuel, Alcorcón, Madrid, Spain.

If we made the change

$$x = \frac{a - u}{1 + au},$$

then

$$dx = \frac{- (1 + au) - (a - u)a}{(1 + au)^2} du = - \frac{1 + a^2}{(1 + au)^2} du,$$

$$1 + ax = 1 + \frac{a(a - u)}{1 + au} = \frac{1 + a^2}{1 + au} ,$$

and

$$1 + x^2 = 1 + \left( \frac{a - u}{1 + au} \right)^2 = \frac{1 + 2au + a^2 u^2 + a^2 - 2au + a^2}{(1 + au)^2} = \frac{(1 + a^2)(1 + u^2)}{(1 + au)^2}. $$

Also,

$$x = 0 \implies u = a,$$

$$x = a \implies a(1 + au) = a - u \implies u = 0.$$

Thus

$$\int_0^a \frac{\log(1 + ax)}{1 + x^2} dx = \int_0^a \frac{\log[(1 + a^2)/(1 + au)]}{1 + u^2} (-du)$$

$$= \int_0^a \frac{\log(1 + a^2)}{1 + u^2} du - \int_0^a \frac{\log(1 + au)}{1 + u^2} du$$

$$= \log(1 + a^2) \arctan a - \int_0^a \frac{\log(1 + au)}{1 + u^2} du,$$

so

$$\int_0^a \frac{\log(1 + ax)}{1 + x^2} dx = \frac{1}{2} \log(1 + a^2) \arctan a.$$ 

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIA BLASKOV, Technical University, Gabrovo, Bulgaria; LEN BOS, University of Calgary; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALThER JANOUS, Ursulinegymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa,
Poland; KEE-WAI LAU, Hong Kong; JOHN H. LINDSEY, Northern Illinois University, DeKalb; BEATRIZ MARGOLIS, Paris, France; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one partial solution received.

The proposer remembered hearing the integral from his calculus professor in Taiwan in 1961. Hess found the integral in Gradshteyn and Ryzhik, p. 556, no. 4.29.18. Janous located it in the Russian book Prudnikov, Brychkov, Marichev, Integrals and Series (Elementary Functions), p. 506, item 2.6.11.1.

\[
\begin{align*}
* & \quad * & \quad * & \quad * & \quad * \\

In \( n \)-dimensional space it is possible to arrange \( n + 1 \) \( n \)-dimensional solid spheres of unit radius in such a way that they all touch one another. Determine the radius of the small solid sphere that touches all \( n + 1 \) of these spheres.

Solution by John H. Lindsey, Northern Illinois University, DeKalb.

The \((n + 1)\)-tuples

\[
(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)
\]

are mutually distance \( \sqrt{2} \) apart and really lie in \( \mathbb{R}^n \) since they lie in the hyperplane \( x_1 + x_2 + \cdots + x_{n+1} = 1 \). The center of mass

\[
\left( \frac{1}{n+1}, \ldots, \frac{1}{n+1} \right)
\]

is distance

\[
\sqrt{\left(1 - \frac{1}{n+1}\right)^2 + n \left(\frac{1}{n+1}\right)^2} = \sqrt{\frac{n}{n+1}}
\]

from the original points. Blowing up the scale by a factor \( \sqrt{2} \) to make the original \( n + 1 \) points mutually distance 2 apart, this becomes \( \sqrt{2n}/(n + 1) \) so the center ball should have radius

\[
\sqrt{\frac{2n}{n+1}} - 1.
\]

[Lindsey ended by noting that the result is known and referring to a book of C.A. Rogers, probably Packing and covering, Cambridge Univ. Press, 1964, p. 79. —Ed.]

Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; CARLES ROMERO CHESA, I.B. Manuel Blancafort, La Garriga, Catalonia, Spain; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

\[
\begin{align*}
* & \quad * & \quad * & \quad * & \quad *
\end{align*}
\]

Let \( \triangle ABC \) be a triangle with circumradius \( R \) and area \( F \), and let \( P \) be a point in the same plane. Put \( AP = R_1, BP = R_2, CP = R_3 \), \( R' \) the circumradius of the pedal triangle of \( P \), and \( p \) the power of \( P \) relative to the circumcircle of \( \triangle ABC \). Prove that

\[
18R^2 R' \geq a^2 R_1 + b^2 R_2 + c^2 R_3 \geq 4F \sqrt{3|p|}.
\]

Combination of solutions of Murray S. Klamkin, University of Alberta, and the proposer.

By using the following known properties [1] of the pedal triangle,

\[
\text{sides (} a', b', c' \text{)} = \left( \frac{aR_1}{2R}, \frac{bR_2}{2R}, \frac{cR_3}{2R} \right),
\]

\[
\text{area } F' = \frac{|R^2 - OP^2| F}{4R^2} = \frac{|p|F}{4R^2},
\]

the proposed inequalities can be written in the more revealing form

\[
9RR' \geq a'a' + b'b' + c'c' \geq 4\sqrt{3FF'}.
\]

(1)

(It is to be noted that the formula for \( F' \) is derived in [1] assuming \( P \) is an interior point. However, it can be shown that if \( P \) is an exterior point then this formula remains valid.)

We now show that (1) is valid for any two arbitrary triangles of sides \((a, b, c)\) and \((a', b', c')\). The right hand inequality is just Crux 1114 [1987: 185]. For the left hand inequality, we use the known inequality \( 9R^2 \geq a^2 + b^2 + c^2 \) (see [2]) to give, by Cauchy’s inequality,

\[
3R \cdot 3R' \geq \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{a'^2 + b'^2 + c'^2} \geq a'a' + b'b' + c'c',
\]

with equality if and only if both triangles are equilateral.

References:

* * * * *


Consider a “signed harmonic series”

\[
\sum_{n=1}^{\infty} \frac{\epsilon_n}{n}, \quad \epsilon_n = \pm 1 \text{ for } n.
\]

Assuming that plus and minus signs occur with equal frequency, i.e.

\[
\lim_{n \to -\infty} \frac{\epsilon_1 + \cdots + \epsilon_n}{n} = 0,
\]

prove or disprove that the series necessarily converges.
I. Solution by Walther Janous, Ursulengymnasium, Innsbruck, Austria.
The answer is negative. Indeed, let $P$ be the set of primes. Then consider

$$\mathbb{N}\setminus P = \{1, 4, 6, 8, 9, 10, 12, \ldots \} = \{ n_1, n_2, n_3, \ldots \}.$$ 

As an alternating series, $\sum_{k=1}^{\infty} (-1)^k / n_k$ converges, whereas $\sum_{p \in P} 1/p$ diverges. So, putting $e_{n_k} = (-1)^k$ for all $k = 1, 2, \ldots$ and $e_p = 1$ for all $p \in P$,

$$\sum_{n=1}^{\infty} \frac{e_n}{n} = \sum_{p \in P} \frac{1}{p} + \sum_{k=1}^{\infty} (-1)^k \frac{1}{n_k}$$ 

diverges. But

$$\frac{\pi(n) - 1}{n} \leq \frac{\epsilon_1 + \cdots + \epsilon_n}{n} \leq \frac{\pi(n)}{n},$$

where $\pi(n)$ is the number of primes $\leq n$. Since $\pi(n) \sim n / \log n$, we infer

$$\lim_{n \to \infty} \frac{\epsilon_1 + \cdots + \epsilon_n}{n} = 0.$$

[Editor’s note: the above proof refers to the Prime Number Theorem ($\pi(n) \sim n / \log n$), but only needs the consequence $\pi(n)/n \to 0$.]

II. Generalization by Robert B. Israel, University of British Columbia.

It does not necessarily converge. More generally, the following is true.

THEOREM. Consider any sequence $a_n \geq 0$ such that $\sum_{n=1}^{\infty} a_n = \infty$. Then there is a sequence $(\epsilon_n)$ such that $\epsilon_n = \pm 1$,

$$\lim_{n \to \infty} \frac{\epsilon_1 + \cdots + \epsilon_n}{n} = 0, \quad \text{and} \quad \sum_{n=1}^{\infty} \epsilon_n a_n \text{ diverges.}$$

Proof. Take a sequence of integers $(t_k)$ such that $t_0 = 0$, and for all $k$,

$$\sum_{n=t_{k+1}}^{t_{k+1}+1} a_n \geq 2k + 1,$$

$$t_{k+1} - t_k \geq 9(t_k - t_{k-1}),$$

and $t_{k+1} - t_k$ is a positive multiple of $2k + 1$. Let

$$S_k = \sum_{n=t_{k+1}}^{t_{k+1}+1} a_n \quad (\geq 2k + 1).$$

We have $S_k = \sum_{j=0}^{2k} T_{jk}$ where

$$T_{jk} = \sum \{ a_n : t_k + 1 \leq n \leq t_{k+1}, \quad n \equiv j \mod 2k + 1 \}. $$
\[ T_{jk} \text{ is the sum of every } (2k + 1) \text{st term of } S_k, \text{ starting at the } j \text{th.} \] Let \( U_k \) be the subset of \( \{0, 1, \ldots, 2k\} \) corresponding to the \( k + 1 \) largest \( T_{jk} \) values (breaking ties arbitrarily). Thus

\[
\sum_{j \in U_k} T_{jk} \geq \frac{k + 1}{2k + 1} S_k.
\]

If \( t_k + 1 \leq n \leq t_{k+1} \), let \( \epsilon_n = +1 \) if \( n \mod 2k + 1 \in U_k \), and \(-1\) otherwise. We have

\[
\sum_{n=t_k+1}^{t_{k+1}} \epsilon_n a_n = \sum_{j \in U_k} T_{jk} - \sum_{j \not\in U_k} T_{jk} \geq \frac{k + 1}{2k + 1} S_k - \frac{k}{2k + 1} S_k = \frac{S_k}{2k + 1} \geq 1,
\]

so that the series \( \sum \epsilon_n a_n \) diverges. Next we will estimate \( \sum_{n=1}^{N} \epsilon_n \). Note that in any \( 2k + 1 \) consecutive \( n \)'s from \( t_k + 1 \) to \( t_{k+1} \) there will be \( k + 1 \) positive and \( k \) negative \( \epsilon_n \), adding to \( 1 \). Thus

\[
\sum_{n=t_k+1}^{t_{k+1}} \epsilon_n = \frac{t_{k+1} - t_k}{2k + 1}.
\]

On the other hand, if \( t_k + 1 \leq N \leq t_{k+1} \), we have \( N = t_k + q(2k + 1) + r \) for some \( q \) and \( r \) with \( 0 \leq r < 2k + 1 \). Then

\[
\left| \sum_{n=1}^{N} \epsilon_n \right| \leq \sum_{j=0}^{k-1} \left| \sum_{n=t_k+1}^{t_{k+1}} \epsilon_n \right| + \sum_{i=0}^{q-1} \left| \sum_{n=t_k+i(2k+1)+1}^{t_k+i(2k+1)+2k} \epsilon_n \right| + \sum_{n=t_k+q(2k+1)+1}^{N} \epsilon_n
\]

\[
\leq \sum_{j=0}^{k-1} \frac{t_{j+1} - t_j}{2j + 1} + q + r.
\]

Note that for \( j \leq k - 1 \),

\[
\frac{t_{j+1} - t_j}{2j + 1} \leq \frac{3^j - 2k + 2(t_k - t_{k-1})}{2j + 1} \leq \frac{3^j}{2k - 1} - \frac{3^{j-k} - 2k + 2(t_k - t_{k-1})}{2k - 1}.
\]

since \( 3j/(2j + 1) \) is an increasing function of \( j \). This implies

\[
\left| \frac{1}{N} \sum_{n=1}^{N} \epsilon_n \right| \leq \frac{3(t_k - t_{k-1})}{(2k - 1)N} \sum_{j=0}^{k-1} \left( \frac{1}{3} \right)^{k-j} + \frac{q}{N} + \frac{r}{N}
\]

\[
< \frac{3}{2(2k - 1)} \frac{t_k - t_{k-1}}{N} + \frac{q}{q(2k + 1)} + \frac{2k + 1}{t_k - t_{k-1}}
\]

\[
\rightarrow 0 \text{ as } n \rightarrow \infty,
\]

since \( N > t_k \geq t_k - t_{k-1} \geq 9^{k-1} \) for all \( k \).

\textbf{Also solved by MURRAY S. KLAMKIN, University of Alberta; JIAQI LUO, Cornell University, Ithaca, N.Y.; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one incorrect solution submitted.}
The proposer’s solution was the same as Janous’s. The other three solutions used the fact that the series $\sum (n \log n)^{-1}$ diverges.

* * * * *

Let $M$ be the midpoint of $BC$ of a triangle $ABC$. Suppose that $\angle BAM = \angle C$ and $\angle MAC = 15^\circ$. Calculate angle $C$.

I. Solution by Jordi Dou, Barcelona, Spain.
Let $\Omega$ be the circle through $AMC$, with centre $O$ and radius $R$. Since $\angle MAC = 15^\circ$, $\angle MOC = 30^\circ$. Since $\angle MCA = \angle MAB$, $\angle B$ will be tangent to $\Omega$ at $A$. The distance $MM'$ of $M$ to $AC$ is $R \sin 30^\circ = R/2$, and since $CB = 2CM$ the distance $BB'$ of $B$ to $AC$ will be $R = AO$. Thus $AB$ is parallel to $OC$, so $\angle AOC = 90^\circ$ and $\angle C = \angle BAM = \angle BAC - \angle MAC = 45^\circ - 15^\circ = 30^\circ$.

II. Solution by Dag Jonsson, Uppsala, Sweden.
Let $a = \angle C = \angle BAM$, $c = |AB|$ and $a = |BM| = |MC|$. Since $\Delta ABC$ is similar to $\Delta MBA$, $c/(2a) = a/c$, i.e. $c = \sqrt{2a}$. The sinus theorem applied to $\Delta ABC$ gives

$$\frac{2a}{\sin(a + 15^\circ)} = \frac{\sqrt{2a}}{\sin a},$$

so

$$\frac{\sin(a + 15^\circ)}{\sin a} = \frac{1/\sqrt{2}}{1/2} = \frac{\sin(30^\circ + 15^\circ)}{\sin 30^\circ},$$

with the obvious solution $a = 30^\circ$. It is unique since

$$\frac{\sin(a + 15^\circ)}{\sin a} = \cos 15^\circ + \frac{\sin 15^\circ}{\tan a}$$

is decreasing in $a$ (here $a < 90^\circ$).

Also solved by HAYO AHLBURG, Benidorm, Spain; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; MARGARET IZIENICKI, Mount Royal College, Calgary; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; MARIA MERCEDES SÁNCHEZ BENITO, I.B. Luis Bunuel, Madrid, Spain; K.R.S. SASTRY, Addis Ababa, Ethiopia;
Janous and Walker gave generalizations.

* * * * * *


Determine sharp upper and lower bounds for the sum of the squares of the sides of a quadrilateral with given diagonals $e$ and $f$. For the upper bound, it is assumed that the quadrilateral is convex.

Solution by Hayo Ahlburg, Benidorm, Spain.

The required upper and lower bounds are respectively

$$2(e^2 + ef + f^2) \text{ and } e^2 + f^2.$$  

In the quadrilateral $ABCD$ where $U, V$ are the midpoints of the diagonals $AC = e$ and $BD = f$ and the point of intersection of the diagonals is $I$, we have the relationship (attributed by Carnot to Euler)

$$(AB)^2 + (BC)^2 + (CD)^2 + (DA)^2 = e^2 + f^2 + 4(UV)^2$$

(see N. Altshiller Court, *College Geometry*, pp. 126 and 300). The lower bound $e^2 + f^2$ is reached for $UV = 0$, $U \equiv V \equiv I$. $ABCD$ is then a parallelogram. The upper bound is approached, not reached, if $UV$ grows as far as possible for given values of $e$ and $f$.

Looking at $\triangle UVI$, we see that

$$UV < UI + IV < UA + DV = \frac{e + f}{2},$$

so that

$$(AB)^2 + (BC)^2 + (CD)^2 + (DA)^2 < 2(e^2 + ef + f^2).$$

As $A$ and $D$ get closer and $\angle UV$ approaches $180^\circ$, we get as close as we want to this upper bound. It would only be reached by a degenerate "quadrilateral", a straight line $BVAIDUC$, with $A \equiv I \equiv D$.

Also solved by C. Festraets-Hamoir, Brussels, Belgium; John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta; Marcin E. Kuczma, Warszawa, Poland; P. Penning, Delft, The Netherlands; and the proposer. One incorrect solution was received.

The proposer notes that the lower bound provides an immediate proof of a conjecture of Tutescu, referred to on p. 411 of D.S. Mitrinovic et al, Recent Advances in Geometric Inequalities.

* * * * * *
The rational number $\frac{5}{2}$ has the property that when written in decimal expansion, i.e. $2.5$ (2.5 in North America), there appear exactly the (base 10) digits of numerator and denominator in permuted form. Do there exist infinitely many $m, n \in \mathbb{N}$, neither ending in 0, so that $m/n$ has the same property?

I. Solution by Robert B. Israel, University of British Columbia.

Yes. The easiest infinite family of solutions is

$$\frac{59}{2} = 29.5, \quad \frac{599}{2} = 299.5, \quad \frac{5999}{2} = 2999.5, \quad \cdots,$$

i.e., $(9)_k/2 = 2(9)_k.5$, $k = 1, 2, 3, \cdots$. Here $(b)_k$ denotes a block of digits $(b)$ repeated $k$ times. For convenience I’m using $ab$ to denote the concatenation of blocks of digits $a$ and $b$; I’ll use $\times$ for multiplication. There are also infinitely many solutions with $n = 5$, e.g.,

$$\frac{6028124(9)_k}{5} = 1205624(9)_k.8.$$

The real question, I think, is whether there are solutions for infinitely many different $n$. I conjecture that for every odd positive integer $t$ there are infinitely many solutions for $n = 2^t$. (It is not hard to show that there are no solutions for $n = 5^t$ with $t > 1$, because $m/n$ will have fewer digits than required.)

Some examples with $n = 2^t$, $t$ odd:

$$t = 3 : \quad \frac{22514(285714)_k}{8} = 2814(285714)_k.25$$

$$t = 5 : \quad \frac{7052}{32} = 220.375$$

$$t = 7 : \quad \frac{20516}{128} = 160.28125$$

$$t = 9 : \quad \frac{6983312}{512} = 13639.28125$$

$$t = 11 : \quad \frac{3781162256}{2048} = 1846270.6328125$$

Two partial results:

LEMMA 1. There are no solutions for $n = 2^t$ where $t$ is even.

Proof. Write the equation as $m/n = x/10^t$ where $x$ is an integer containing precisely the digits of $m$ and $n$. Since each integer is congruent mod 9 to the sum of its digits, we would have $m + n \equiv x$ as well as $m \equiv n \times x$, so that

$$\quad (n - 1) \times x = n \times x - x \equiv m - (m + n) = -n \text{ mod } 9.$$

But if $n = 2^t$ with $t$ even, $n - 1$ is divisible by 3 while $n$ is not, so this is impossible. \qed
LEMMA 2. If there is at least one solution \((m, n)\) for a given \(n = 2^t\) with \(m \geq n + 8\), then there are infinitely many.

Proof. Consider a solution 
\[
\frac{A b}{n} = \frac{X}{10^\ell},
\]
where \(A\) is an integer and \(b\) a single nonzero digit, and \(X\) is an integer containing precisely all digits of \(A, n\) and \(b\) (we could get along without the \(b\) except for the rather artificial restriction that \(m\) should not end in \(0\)). Let \(A = n \times Y + r\) where \(0 \leq r < n\). Then
\[
X = 10^\ell \times \frac{10 \times A + b}{n} = 10^{\ell+1} \times Y + 10^\ell \times \frac{10 \times r + b}{n}
\]
\[
= YZ \quad \text{where } Z = 10^\ell \times \frac{10 \times r + b}{n} < 10^{\ell+1} \text{ has } \ell + 1 \text{ digits.}
\]
Then for a nonnegative integer \(C\) of \(d\) digits (with leading zeros allowed), we will obtain a new solution 
\[
\frac{AC b}{n} = \frac{YC Z}{10^\ell},
\]
i.e.
\[
\frac{10^{d+1} \times A + 10 \times C + b}{n} = 10^{d+1} \times Y + 10 \times C + Z \times 10^{-\ell},
\]
if
\[
(10^d - 1) \times r = (n - 1) \times C.
\]
(1)
Take \(d > 0\) so that \(10^d \equiv 1 \mod (n - 1)\) (\(d\) exists because \((10, 2^t - 1) = 1\) when \(t\) is odd), and let
\[
C = r \times \frac{10^d - 1}{n - 1};
\]
since \(0 \leq r < n\), \(C < 10^d\), so this \(C\) will have at most \(d\) digits, and (1) is satisfied.

The condition \(m \geq n + 8\) ensures that \(YCZ\) has no leading zeros. For, if \(Ab = m < 10 \times n\) we get \(Y = 0\) and \(r = A\) and the new solution would be
\[
\frac{ACb}{n} = \frac{CZ}{10^\ell}.
\]
But then (since \(m = Ab = rb = 10 \times r + b\))
\[
r = \frac{m - b}{10} \geq \frac{n - 1}{10}
\]
so that
\[
C = r \times \frac{(10^d - 1)}{n - 1} \geq \frac{10^d - 1}{10}.
\]
Since \(C\) is an integer, this means that \(C \geq 10^{d-1}\) and there are no leading zeros in \(C\). \(\Box\)

[Editor’s note. The condition “neither \(m\) nor \(n\) ends in 0” was meant by the editor to prevent such trivial solutions as \(50/2 = 25.0, 500/2 = 250.0, \cdots\) and \(5/20 = 0.25, 5/200 = 0.025, \cdots\). There was probably a better way to do this.]
II. Solution by Richard K. Guy, University of Calgary.
Rather trivially: $5/2, 59/2, 599/2, 5999/2, \cdots$.
If recurring decimals are allowed:
\[
\frac{8572}{3} = 2857.3, \quad \frac{8571428572}{3} = 2857142857.3,
\]
\[
\frac{8571428571428572}{3} = 2857142857142857.3, \cdots,
\]
also
\[
\frac{816639344262295081967}{123} = 6639344262295081967.21138,
\]
\[
\frac{816639344262295081967213}{123} = 6639344262295081967213.11382
\]
(what is the next member of this sequence?),(does this belong to an infinite family of solutions?), and so on.

Except for one reader who misinterpreted the problem, no other solutions were sent in. Readers can make amends by settling Israel’s conjecture or by finding other “repeating decimal” solutions à la Guy!


* * * * *

Circles ,1 and ,2 have a common chord $PQ$. $A$ is a variable point of ,1. $AP$ and $AQ$ intersect ,2 for the second time in $B$ and $C$ respectively. Show that the circumcentre of $\triangle ABC$ lies on a fixed circle. (This problem is not new. A reference will be given when the solution is published.)

I. Solution by José Yusty Pita, Madrid, Spain.
Let $N$ be the midpoint of $BC$ and $M$ the circumcenter of $\triangle ABC$. $M$ belongs to the perpendicular bisector of $BC$, so
\[
MN = NB \cdot \cot A.
\]
But $NB = BC/2$ is constant because, with $O_2$ the center of ,2,
\[
\angle A = \angle BQC - \angle PBQ = \frac{\angle BO_2C - \angle PO_2Q}{2},
\]
and $\angle A$ and $\angle P O Q$ are constant. Therefore $M N$ is constant, and $O_2 N$ is also constant (since $B C$ is). So $M O_2 = M N - O_2 N$ is constant and the locus of $M$ is a circle concentric with $O_2$.

II. Solution by C. Fresnachts-Hamoir, Brussels, Belgium.

Soit $K$ le centre du cercle circonscrit à $A B C$ et soient $O_1, O_2$, les centres respectifs de $\Gamma_1$ et $\Gamma_2$. L’inversion de pol $A$ et de puissance $A P \cdot A B = A Q \cdot A C$ (i) transforme le cercle $\Gamma_1$ en la droite $B C$, d’où $A O_1 \perp B C$, et (ii) transforme le cercle circonscrit au triangle $A B C$ en la droite $P Q$, d’où $A K \perp P Q$. On a ainsi $A O_1 || K O_2$ et $A K || O_1 O_2$, donc $A O_1 O_2 K$ est un parallélogramme et $\overrightarrow{A K} = O_1 O_2$, quelle que soit la position de $A$ sur $\Gamma_1$. Le lieu de $K$ est l’image du cercle $\Gamma_1$ par la translation $O_1 O_2$.

Also solved by JORDI DOU, Barcelona, Spain (two solutions); WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

Most solvers observed that the circle on which the circumcenter of $\triangle A B C$ lies is congruent to $\Gamma_1$. This nice result is in Solution II, but it seems (?) it can’t be deduced from Solution I. Of Dou’s two solutions, his first was like Solution I, and his second gave the same conclusion as Solution II, with some extra properties of the diagram as well.

The proposer found the problem (without solution) in Aref and Wernick, Problems and Solutions in Euclidean Geometry, Dover, New York, Exercise 6.17.

* * * * *


Suppose $\alpha, \beta, \gamma$ are arbitrary angles such that $\cos \alpha \neq \cos \beta$, and $x$ is a real number such that

$$x^2 \cos \beta \cos \gamma + x(\sin \beta + \sin \gamma) + 1 = 0$$

and

$$x^2 \cos \gamma \cos \alpha + x(\sin \gamma + \sin \alpha) + 1 = 0.$$ 

Prove that

$$x^2 \cos \alpha \cos \beta + x(\sin \alpha + \sin \beta) + 1 = 0.$$ 


Note that $C$ has to be eliminated from the first two equations. This is simple with the aid of the “trick” of solving first for $\sin C$ and $\cos C$. By subtracting, the equations give

$$x(\cos A - \cos B) \cos C = - (\sin A - \sin B);$$ (1)
by multiplying the first equation by \( \cos A \) and the second by \( \cos B \) and subtracting, they give

\[
x(\cos A - \cos B) \sin C = -(\cos A - \cos B) + x \sin(A - B).
\]

Note that both results become identities for \( A = B \). Introduce

\[
s = \frac{A + B}{2}, \quad v = \frac{A - B}{2},
\]

and use

\[
-2 \sin s \sin v = \cos A - \cos B, \quad 2 \cos s \sin v = \sin A - \sin B;
\]

then (1) and (2) become, after division by \( \sin v \),

\[
\begin{align*}
x \sin s \cos C &= \cos s, \\
x \sin s \sin C &= -\sin s - x \cos v.
\end{align*}
\]

Elimination of \( C \) is now simple by squaring and adding:

\[
x^2 \sin^2 s = x^2 \sin^2 s(\cos^2 C + \sin^2 C) \\
&= \cos^2 s + \sin^2 s + 2x \sin s \cos v + x^2 \cos^2 v \\
&= 1 + 2x \sin s \cos v + x^2 \cos^2 v,
\]

or

\[
x^2(\cos^2 v - \sin^2 s) + 2x \sin s \cos v + 1 = 0,
\]

or

\[
x^2 \cos(v + s) \cos(v - s) + x(\sin(s + v) + \sin(s - v)) + 1 = 0.
\]

Introducing \( A \) and \( B \) again yields the desired result.

Also solved by HAYO AHLBURG, Benidorm, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; G.P. HENDERSON, Campbeltown, Ontario; JOHN H. LINDSEY, Ft. Myers, Florida; and JEAN-MARIE MONIER, Lyon, France.

Bellot notes that the problem appears (without solution) in Hobson’s Plane Trigonometry, Fifth Edition, Cambridge Univ. Press, 1921, Chapter 6, Example 11, p. 95.

\[
\begin{array}{cccc}
* & * & * & * \\
\end{array}
\]


For each fixed positive real number \( a_n \), maximise

\[
\frac{a_1a_2\ldots a_{n-1}}{(1+a_1)(a_1+a_2)(a_2+a_3)\ldots(a_{n-1}+a_n)}
\]

over all positive real numbers \( a_1, a_2, \ldots, a_{n-1} \).
Solution by Chris Wildhagen, Rotterdam, The Netherlands.

Let

\[ P = \frac{a_1 a_2 \ldots a_{n-1}}{(1 + a_1)(a_1 + a_2)\ldots(a_{n-1} + a_n)}. \]

\( P \) can be rewritten as \( P = Q^{-1} \), where

\[ Q = (1 + b_1)(1 + b_2)\ldots(1 + b_n), \]

with \( b_1 = a_1 \) and \( b_i = a_i/a_{i-1} \) for \( 2 \leq i \leq n \). Note that \( b_1 b_2 \ldots b_n = a_n \). We have to minimize \( Q \). To achieve this, we use the following [Holder’s] inequality:

\[ \sqrt[n]{(u_1 + v_1)(u_2 + v_2)\ldots(u_n + v_n)} \geq \sqrt[n]{u_1 u_2 \ldots u_n} + \sqrt[n]{v_1 v_2 \ldots v_n}, \]

where \( u_i, v_i > 0 \) for all \( i \) and equality holds if and only if

\[ \frac{u_1}{v_1} = \frac{u_2}{v_2} = \cdots = \frac{u_n}{v_n}. \]

Applying this result gives

\[ \sqrt[n]{Q} \geq 1 + \sqrt[n]{b_1 b_2 \ldots b_n} = 1 + a_n^{1/n}, \]

hence

\[ Q \geq (1 + a_n^{1/n})^n, \]

with equality if and only if \( b_1 = b_2 = \cdots = b_n = a_n^{1/n}, \) i.e.,

\[ a_i = a_n^{i/n}, \quad 1 \leq i \leq n - 1. \quad (1) \]

We can conclude that

\[ P \leq (1 + a_n^{1/n})^{-n}, \]

with equality if and only if (1) holds.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; RICHARD I. HESS, Rancho Palos Verdes, California; WALThER JANOUS, Ursulinen Gymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; and the proposers. Two other readers sent in upper bounds for the given expression which were not attained for all values of \( a_n \).

Klamkin gave a generalization.

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